CHIRAL AND HEAVY-QUARK EFFECTIVE THEORIES

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In these lectures, some of the basic ideas of effective field theories in particle physics are presented. These ideas are applied to two situations of current interest: the $1/m_Q$ expansion and its application to b and c systems, and chiral perturbation theory and its application to the light mesons.

1 Effective Field Theories: Philosophy and Examples

1.1 Introduction

These lectures introduce the idea of effective field theories (EFT's) as applied to particle physics, and illustrate their application to two areas of current phenomenological interest: heavy-quark effective theory (HQET) and its application to b and c systems, and chiral perturbation theory (χPT)and its application to the light mesons. There is an extensive literature on both these subjects, and I will not be able to come close to covering all the important results in the fields. Rather, I want to focus on the more general aspects of the EFT approach, and stress the ways in which it allows one to exploit a few powerful physical concepts: symmetries, power counting and scaling.

The idea behind an effective field theory is very simple and intuitive: it is simplest to describe physics at a given scale in terms of the relevant degrees of freedom at that scale. Thus, if one is interested in fluid mechanics, it isn't reasonable to attempt to solve for the behaviour of the individual atoms making up the fluid. Similarly, it is not necessary to know about the t quark, let alone M-theory, to do atomic physics to any reasonable level of accuracy. In general, if one is interested in a process typified by some energy scale E, the physics can be described only in terms of the degrees of freedom and interactions relevant at that scale. The effects of new degrees of freedom at larger energy scales Λ_i may then be systematically taken into account by modifying the interactions of the low-energy degrees of freedom accordingly, as an expansion in powers of

$$E/\Lambda_i.$$

The practical benefit of this approach is that choosing the degrees of freedom appropriate to the problem of interest simplifies your life enormously.

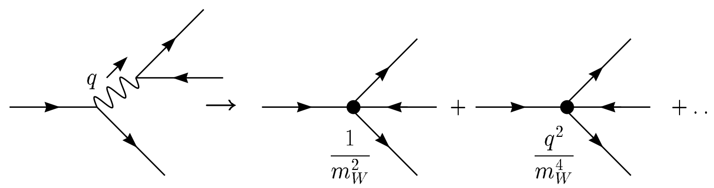

Figure 1: At energies $\ll m_W$, W exchange may be effectively treated as a series of four-fermi interactions.

Hard calculations become easy, and impossible calculations become doable. The essential simplification arises because the various distance scales in a problem are automatically disentangled in this approach *before* the calculation is performed, at the level of the Lagrangian.

Example: Four-Fermi Theory

The most familiar example of an effective field theory is four-fermi theory. Consider the Standard Model at energies much less than m_W. At these energies the W boson cannot propagate, and according to the philosophy above, should not be included as a degree of freedom in the theory. Rather, it should be removed from the theory (or "integrated out"), and its effects taken into account by modifying the interactions of the relevant degrees of freedom. At tree level (we will discuss loops shortly), it is easy to see how to do this. For example, for muon decay at leading order in the weak interactions, the relevant matrix element is given by the diagram in Fig. 1

$$\begin{aligned} i\mathcal{M} &= \frac{ie^2}{8\sin^2\theta_W}\frac{1}{q^2-m_W^2}\bar{\nu}_\mu\gamma^\alpha(1-\gamma_5)\mu\bar{e}\gamma_\alpha(1-\gamma_5)\nu_e \\ &= -\frac{ie^2}{8\sin^2\theta_W m_W^2}\bar{\nu}_\mu\gamma^\alpha(1-\gamma_5)\mu\bar{e}\gamma_\alpha(1-\gamma_5)\nu_e \\ &\qquad \times\left[1+\frac{q^2}{m_W^2}+\frac{q^4}{m_W^4}+\dots\right], \end{aligned} \tag{1}$$

where q is the momentum transfer through the W.

For $q^2 \ll m_W^2$ (which is certainly the situation for muon decay) this amplitude may be reproduced to *arbitrary* accuracy by a series of local operators in an effective Lagrangian,

$$\mathcal{L}_{\rm eff} = \frac{c_1}{m_W^2}\bar{\nu}_\mu\gamma^\alpha(1-\gamma_5)\mu\bar{e}\gamma_\alpha(1-\gamma_5)\nu_e$$

$$+\frac{c_2}{m_W^4}\bar{\nu}_\mu\left(iD-i\overleftarrow{D}\right)^2\gamma^\alpha(1-\gamma_5)\mu\bar{e}\gamma_\alpha(1-\gamma_5)\nu_e+\ldots \qquad (2)$$

where the c_i's are dimensionless, and the ellipses denote additional operators of increasing dimension, suppressed by the appropriate power of $1/m_W^2$. The leading operator is just the usual four-fermi theory of weak interactions, but by keeping a finite number of additional operators this may be made accurate to any fixed order in q^2/m_W^2.

Thus, we have gone from a fairly simple renormalizable field theory to a nonrenormalizable theory with, in principle, an infinite number of terms in the Lagrangian. Furthermore, while the original theory was consistent for all momenta, the effective theory only makes sense for momenta less than m_W. This does not appear to be progress. Nevertheless, an effective Lagrangian such as (2) is actually *much* simpler to work with at low energies than the full theory. There are a number of reasons for this.

- Calculational simplicity: Loop integrals quickly become intractable when there are a number of scales in the problem. In the EFT the scale m_W only appears trivially, in the coefficients of nonrenormalizable operators. This simplifies perturbative calculations tremendously.[a]

- Improved convergence of perturbation theory: Perturbation theory is notoriously poorly behaved in theories with widely disparate scales. For example, let us instead consider inclusive nonleptonic b quark decay,

$$b\to c\bar{u}d+\text{anything}.$$

  Although this process is calculable and free of infrared divergences, at n loops the decay rate contains terms proportional to

$$\left(\frac{\alpha_s(m_b)}{\pi}\right)^n\log^n\frac{m_W^2}{m_b^2} \qquad (3)$$

  due to the presence of two scales, m_b and m_W, in the loop integral. Even though $\alpha_s(m_b)$ is small, the large logarithm can compensate for this, and spoil the convergence of perturbation theory. This result is generic: perturbation theory for problems with different energy scales contains factors of the logarithm of the energy scales. If these are widely separated, perturbation theory will break down, even at scales where the theory is weakly coupled.

[a] A similar comment holds for nonperturbative lattice calculations: simulating the full theory requires lattice spacings on the order of $1/m_W$, whereas the effective theory may be simulated with a much larger lattice spacing, of the order of $1/q$.

In an EFT, the scale m_W does not enter into loop integrals, and no such logarithms arise in matrix elements. Instead, a loop integral proportional to $\alpha_s \log m_W/m_b$ in the full theory is proportional to $\alpha_s \log \mu/m_b$ in the EFT, where μ is either the cutoff or the renormalization scale, depending on whether a cutoff or dimensional regularization is used. Since the renormalization scale is arbitrary, choosing $\mu \sim m_b$ minimizes the logarithms in perturbation theory. More precisely, as we will discuss, one starts with a renormalization scale $\mu \sim m_W$, and then uses the renormalization group equations to lower this to $\mu \sim m_b$. The renormalization group equation sums the series of large logarithms (3) to all orders, incorporating it into the coupling constants of the EFT.

- Dimensional analysis: Dimensional analysis can be a powerful tool to extract physical information with a minimal amount of work. In four-fermi theory, for example, since the leading four-fermi operator is dimension six, the effects of the weak interactions manifestly scale like p^2/m_W^2, where p is the typical momentum scale in the problem. We will see less trivial examples of the power of dimensional analysis a number of times in these lectures.

- Manifest symmetries: Since the weak interactions are parity violating, parity is not even an approximate symmetry of the electroweak Lagrangian. Nonetheless, at low energies parity is approximately conserved. In the EFT, this is manifest: parity is an *exact* symmetry of all operators with dimension less than six. By dimensional analysis, this immediately tells us that parity violating effects are suppressed at low energies by p^2/m_W^2. By contrast, in the full theory parity is violated in the renormalizable interactions, and the scaling of parity violating effects is not manifest. Once again, this is a generic result. In an EFT, approximate symmetries at low energies are exact symmetries of the theory at leading order. Symmetry breaking terms suppressed by powers of p^2/Λ^2 (where Λ is the scale of new physics) arise from nonrenormalizable operators, and the size of such effects is just given by dimensional analysis. This greatly simplifies the analysis of approximate symmetries and symmetry breaking in the theory.

Example: Why is the Sky Blue?

I will close this section with a nice example of the power of dimensional analysis which I learned from Refs. [1,2]: the EFT description of Rayleigh scattering, which explains why the sky is blue. Consider the scattering of low energy light off neutral atoms. To analyze this system from an effective field theory, we need to identify the scales and symmetries of the problem. Identifying the relevant scales will determine the power counting for the problem, while the symmetries will constrain the operators we include in the EFT.

The symmetries of the problem are the obvious ones: Lorentz and gauge invariance. There are a number of relevant scales - the photon energy E_γ, the excitation energy of the atom ΔE, the size of the atom r_0 and the mass m. These scales are well separated, with the hierarchy

$$E_\gamma \ll \Delta E \ll r_0^{-1} \ll m. \tag{4}$$

Given the $E_\gamma \ll m$, there is effectively no recoil, so the four-velocity v^μ of the atom is conserved (we will see more of this in the next lecture). We will describe the atom by a field φ. At low energies, the atom is nonrelativistic; thus it should be described by a nonrelativistic Lagrangian

$$\mathcal{L} = \varphi^\dagger \left(i\partial \cdot v - \frac{\partial^2}{2m} \right) \varphi + \mathcal{L}_{\rm int} \tag{5}$$

(in the rest frame, where $v^\mu = (1, \vec{0})$, this is just $E - p^2/2m$).

The interaction Lagrangian contains, in principle, an infinite number of terms, restricted only by gauge and Lorentz invariance:

$$\mathcal{L}_{\rm int} = c_1 \varphi^\dagger \varphi F_{\mu\nu} F^{\mu\nu} + c_2 \varphi^\dagger \varphi v^\alpha F_{\alpha\mu} v_\beta F^{\beta\mu} + c_3 \varphi^\dagger \varphi (v^\alpha \partial_\alpha) F_{\mu\nu} F^{\mu\nu} + \ldots. \tag{6}$$

To organize these in terms of importance, we need to do some dimensional analysis. The mass dimensions of the various components of $\mathcal{L}$ are

$$[\varphi] = \frac{3}{2}, \quad [F^{\mu\nu}] = 2, \quad [\partial_\mu] = 1 \tag{7}$$

and thus we have

$$[c_1] = [c_2] = -3, \quad [c_3] = -4, \ldots. \tag{8}$$

Since derivatives in the effective theory pick up factors of E_γ, the typical momentum scale in the problem, the effects of the c_3 operator are suppressed by

$$\frac{E_\gamma}{\Delta E}$$

(since ΔE is the next energy scale in the problem) relative to c_1 and c_2. Thus, at low energies we need only keep the first two terms in $\mathcal{L}_{\rm int}$, and we expect corrections to these results of order $E_\gamma/\Delta E$. Finally, since at low energies no internal structure of the atoms can be probed, the only scale which c_1 and c_2

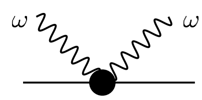

Figure 2: In the EFT, photon-atom scattering occurs through contact interactions.

can be sensitive to is r_o, the size of the atoms, and therefore by dimension analysis

$$c_1,\ c_2 \sim r_0^3. \tag{9}$$

The effective interaction is therefore

$$\mathcal{L}_{\rm int} = r_0^3 \left[a_1 \varphi^\dagger \varphi F_{\mu\nu} F^{\mu\nu} + a_2 \varphi^\dagger \varphi v^\alpha F_{\alpha\mu} v_\beta F^{\beta\mu} \right] + O(E_\gamma/\Delta E) \tag{10}$$

where a_1 and a_2 are unknown coefficients of order 1.

Now let us consider elastic photon-atom scattering in this theory. At tree level, it occurs through the contact diagram in Fig. 2. If the energy of the photon is ω, the amplitude is proportional to

$$r_0^3 \omega^2$$

(since there are two derivatives in both terms in $\mathcal{L}_{\rm int}$). Squaring this up to give the cross section (since phase space is dimensionless), we find

$$\sigma \sim r_0^6 \omega^4. \tag{11}$$

This is Rayleigh's famous ω^4 result for scattering, and explains why the sky is blue: blue light scatters more strongly than red. Conversely, sunsets are red since the higher frequencies are scattered away from the forward direction.

To calculate a_1 and a_2 would require more work (this is known as "matching", and will be the subject of much discussion in these lectures), but the amusing result of this example is that Rayleigh's ω^4 law may be obtained with so little work, simply as a consequence of gauge invariance and power counting.

1.2 Renormalizability vs. Power Counting

Since an EFT contains an infinite number of operators of arbitrarily high dimension, it is clearly not renormalizable. Nonrenormalizable theories were once held in low regard because of their alleged lack of predictive power. However, in an EFT obtained after integrating out physics at a scale M, one retains

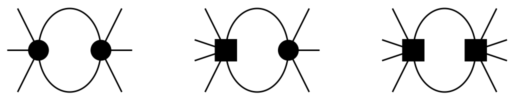

Figure 3: Divergent graphs in φ^5 (+ counterterms) theory. The circle denotes the φ^5 interaction; the box denotes the φ^6 interaction which is required as a counterterm.

predictive power order by order in p/M. To illustrate this, let us consider adding a nonrenormalizable term to φ^4 theory,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{\mu^2}{2}\varphi^2 - \lambda_1\varphi^4 - \lambda_2\varphi^5. \tag{12}$$

The usual argument about the importance of renormalizability arises from simply considering the divergences encountered in such a theory. For example, the graph in Fig. 3 (a) is divergent, proportional to $\varphi^6 \log\Lambda$, where Λ is a UV cutoff. To absorb this divergence, we must introduce a φ^6 counterterm into the theory. Since this is a new parameter in the theory, we have a new coupling constant which must be fixed from experiment. But now graphs such as Fig. 3 (b) and (c) arise, proportional to $\varphi^7 \log\Lambda$ and $\varphi^8 \log\Lambda$ respectively, forcing us to introduce φ^7 and φ^8 counterterms, *ad infinitum*. Thus, the argument runs, a nonrenormalizable field theory has no predictive power since it requires an infinite number of coupling constants, all of which must be fixed by experiment.

While this argument is, strictly speaking, correct, let's look at it from the point of view of dimensional analysis. The nonrenormalizable part of the EFT has the form

$$\mathcal{L}_{\text{int}} \sim \frac{a_1}{M}\varphi^5 + \frac{a_2}{M^2}\varphi^6 + \dots \tag{13}$$

where M is some mass scale chosen such that the a_i's are dimensionless (and generically of order one, as would be the case if there were new physics at the scale M). At low energies, $p \ll M$, the effects of higher dimension operators will be suppressed by powers of p/M, just by dimensional analysis, and are thus, in a technical sense we shall define in a moment, "irrelevant." The higher the dimension of the operator, the smaller its contribution at low energies, at least at tree level. Now let's look at some loop graphs.

Regulating the theory with a cutoff Λ, the graph in Fig. 4 (a) simply renormalizes the φ^5 coupling λ_2,

$$(a) \sim \frac{1}{M}\varphi^5 \log\Lambda + \text{finite}. \tag{14}$$

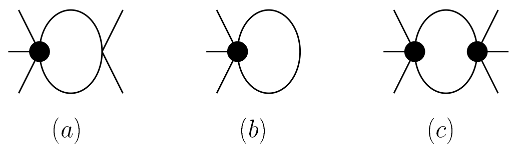

Figure 4: More divergent graphs in φ^5 theory.

Since λ_2 was a free parameter in the first place, this does not reduce the predictive power of the theory. Fig. 4 (b) contains both quadratic and logarithmic divergences,

$$(b) \sim \frac{\Lambda^2}{M}\varphi^3 + \frac{1}{M}\varphi\partial_\mu\varphi\partial^\mu\varphi \log\Lambda + \text{finite}. \tag{15}$$

These renormalize additional operators with dimension ≤ 5. This was to be expected. Even in a renormalizable theory, all operators with dimension 4 or less and consistent with the symmetries are required as counterterms, even if they were not included in the Lagrangian originally. In general, if one is interested in working to order $1/M^n$, all operators permitted by the symmetries with dimension $n+4$ or less must be included. A renormalizable field theory is just the $n = 0$ case. So although the operators in Eq. (15) were not included in Eq. (12), they should have been, since they are permitted by the symmetries of the theory, and their coefficients are renormalized by this graph.

Finally, the troublesome graph in Fig. 4 (c) requires a φ^6 counterterm, but the graph is of order $1/M^2$. To the order we are working, this graph should be neglected, since we are not consistently including effects of $O(1/M^2)$. If we wished to work to this order, we would have to include all operators of dimension ≤ 6, and once again there would be enough counterterms to absorb all the UV divergences in the theory. As long as we work consistently to a given order in $1/M$, only a finite number of operators are required as counterterms in the EFT; thus, we have predictive power. We have traded the somewhat artificial concept of renormalizability for the more physical concept of power counting.

As the dimension of operators we include increases, there will be more and more operators in the EFT, giving it more and more free parameters. This is not necessarily a problem. If the full theory is known, these coefficients may all be computed in terms of the parameters of the full theory. This was done at tree level in the four-fermi example, and will be done to one loop in the next example. On the other hand, if the full theory is not known, or is not perturbative, this indeed limits the predictive power of the theory to a given order p/M. We will see an example of this in the third lecture, when we study chiral perturbation theory.

Note that this hierarchy of scales really "explains" renormalizability in the first place.[3] A general Lagrangian may be divided into terms with dimension $D < 4$, $D = 4$ and $D > 4$,

$$\mathcal{L} = \mathcal{L}_{d<4} + \mathcal{L}_{d=4} + \mathcal{L}_{d>4}. \tag{16}$$

In a weakly-coupled theory, terms with $d < 4$ are known as "relevant" operators - they become more important at lower scales. Particle masses are an example of relevant operators. Terms with $d > 4$ are called "irrelevant." They become less important at low energies, being suppressed by powers of momentum over some mass scale (just by dimensional analysis). An example is a four-fermi operator. Terms with $d = 4$ are called "marginal" - their importance doesn't change as the scale is changed. Note that these classifications may be modified by quantum effects. In a weakly interacting theory, quantum corrections push a marginal operator into the relevant or irrelevant side. For example, gauge couplings are marginal by power counting, but in QCD radiative corrections make the gauge coupling relevant, becoming more important at low energies. In a strongly interacting theory, more dramatic changes can occur - for example, irrelevant operators may be strongly enhanced and made relevant.

Thus, any effective theory will look renormalizable at low energies if there is a large hierarchy of scales. The irrelevant operators give small contributions and may be neglected, and all that is left are the renormalizable terms.

Example: QED With a Heavy Scalar

At this stage, let us illustrate some of these ideas with a concrete example, QED with a Yukawa coupling to a heavy scalar field:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\not{D} - m_e)\psi + \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{M^2}{2}\varphi^2 - g\bar{\psi}\psi\varphi. \tag{17}$$

Suppose we are interested in precision measurements of the electron-photon coupling at low momentum

$$p \sim m_e \ll M. \tag{18}$$

Then we can calculate the effects of the heavy scalar in two ways.

Full Theory Approach: In the full theory, the result is given by the graph in Fig. 5 (a) (as well as the wave function graphs). This looks innocuous enough, but if we are really interested in the full dependence of the result on p and m_e (to leading order in p/M and m_e/M) the results is rather complicated.

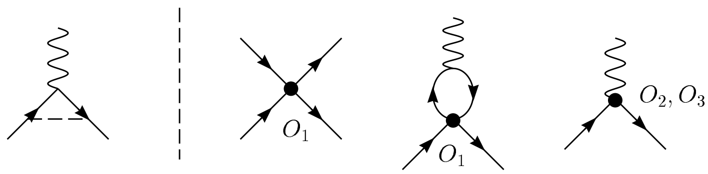

Figure 5: Diagrams contributing to photon-fermion coupling in a theory with a heavy scalar. (a) full theory, (b-d) effective theory.

Straightforwardly evaluating the Feynman integrals results in a several page expression. The graph is complicated because of the presence of several mass scales in the loop integrals. Expanding the result in powers of $1/M$ (again, a nontrivial task given the complicated functional dependence) one eventually finds at leading order

$$\begin{aligned} i\mathcal{M} = & -\frac{m_e g^2}{16\pi^2 M^2} q_\nu \sigma^{\mu\nu} \left[-1 + \log \frac{m_e^2}{M^2} + 2\sqrt{\frac{4m_e^2}{q^2} - 1} \tan^{-1} \sqrt{\frac{4m_e^2}{q^2} - 1} \right] \\ & -\frac{1}{3} \frac{g^2}{16\pi^2 M^2} \left[\not{q} q^\nu - q^2 \gamma^\mu \right] \left[-\frac{5}{6} + \log \frac{m_e^2}{M^2} - \frac{4m_e^2}{q^2} \right. \\ & \left. +2\left(1 + \frac{2m_e^2}{q^2}\right) \sqrt{\frac{4m_e^2}{q^2} - 1} \tan^{-1} \sqrt{\frac{4m_e^2}{q^2} - 1} \right] \\ & +O(1/M^4) \end{aligned} \tag{19}$$

Effective Field Theory Approach: Much of the complication of the full theory calculation is irrelevant, because physics at low energies is insensitive to the form of the loop graph at $p^2 \sim M^2$. The same result may be obtained much more simply in an EFT. For $p, m_e \ll M$, the φ field is not a relevant degree of freedom and should be integrated out of the theory. The effective theory contains only electrons and photons, with the effects of virtual φ exchange included through a series of nonrenormalizable operators O_i:

$$\mathcal{L}_{\text{eft}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} \left(i\not{D} - m_e \right) \psi + \sum_i c_i O_i \tag{20}$$

where the c_i's are determined by demanding that the EFT reproduce the results of the full theory order by order in perturbation theory and p/M. (No new operators are introduced with $d \leq 4$, since the QED Lagrangian contains all operators of dimension ≤ 4 consistent with gauge invariance.)

Let's focus on the dimension six operators, scaling like $1/M^2$. At tree level, the matching is easy. Integrating out the heavy scalar at tree level leads to the four-fermi operator

$$O_1 = \frac{1}{M^2}\bar{\psi}\psi\bar{\psi}\psi \tag{21}$$

shown in Fig. 5(b), with coefficient

$$c_1 = \frac{1}{2}g^2 + O(\alpha). \tag{22}$$

Now, let's look at the effect of this operator on the $ee\gamma$ coupling. The obvious difference between the full and effective theories is that the graph in Fig. 5(c) is divergent, whereas the full theory result is convergent. However, this difference is a short-distance effect which we will absorb into the coupling constants of the low-energy theory. More importantly, the EFT gets the physics right at low loop momenta. Regulating the EFT in a convenient scheme, dimensional regularization in $4-\epsilon$ dimensions, we find after perhaps a page of work,

$$\begin{aligned} i\mathcal{M} = &-\frac{m_e^2 g^2}{16\pi^2 M^2} q_\nu \sigma^{\mu\nu}\left[-\Delta + \log\frac{m_e^2}{\mu^2} + 2\sqrt{\frac{4m_e^2}{q^2}-1}\tan^{-1}\sqrt{\frac{4m_e^2}{q^2}-1}\right] \\ &-\frac{1}{3}\frac{g^2}{16\pi^2 M^2}\left[\not{q}q^\nu - q^2\gamma^\mu\right]\left[-\Delta + \log\frac{m_e^2}{\mu^2} - \frac{5}{3}\right. \\ &\left.-\frac{4m_e^2}{q^2} + 2\left(1+\frac{2m_e^2}{q^2}\right)\sqrt{\frac{4m_e^2}{q^2}-1}\tan^{-1}\sqrt{\frac{4m_e^2}{q^2}-1}\right] \end{aligned} \tag{23}$$

where

$$\Delta \equiv \frac{2}{\epsilon} + \log 4\pi - \gamma_E \tag{24}$$

is the usual divergence arising in dimensional regularization.

Comparing this result with the full theory result, Eq. (19), we see that the EFT has correctly reproduced the nonanalytic dependence on m_e and q, with significantly less effort. This is no accident, of course. Nonanalytic dependence on momenta and masses arises from physics at that scale. Since the EFT gets the physics below the scale M correct, all nonanalytic dependence on scales less than M is correctly reproduced. The only difference between the two results is *analytic* in the external momenta, and can be taken into account in the EFT by adding the local dimension six operators shown in Fig. 5(d),

$$
\begin{aligned}
&c_2(\mu)\frac{m_e}{M^2}\bar\psi\sigma^{\mu\nu}F_{\mu\nu}\psi \quad \text{(electron magnetic moment operator)}\\
&c_3(\mu)\frac{1}{M^2}\bar\psi\partial_\mu F^{\mu\nu}\gamma_\nu\psi \quad \text{(Darwin term)}
\end{aligned}
\tag{25}
$$

where

$$
\begin{aligned}
c_2(\mu) &= -\frac{g^2}{32\pi^2}\left[1+\log\frac{M^2}{\mu^2}-\Delta\right]\\
c_3(\mu) &= -\frac{g^2}{48\pi^2}\left[\frac{5}{6}-\log\frac{M^2}{\mu^2}+\Delta\right].
\end{aligned}
\tag{26}
$$

(The terms containing Δ are usually dropped, with $\overline{\text{MS}}$ being implicit). Note that had the nonanalytic terms not been correctly reproduced, we would have been stymied, since these effects cannot be reproduced by local operators. This would have been a sign that we were working in the wrong effective theory. Note also that the extra ultraviolet divergences in the EFT introduce no problems. The divergences are just absorbed into the matching conditions for c_2 and c_3.

This example illustrates the generalization of the matching procedure already encountered at tree-level to loop graphs. The procedure is generic: calculate an amplitude in the full and effective theories, then add the appropriate operators to the EFT so that it reproduces the scattering amplitudes in the full theory, order by order in the ratio of scales. As long as you are working in the correct EFT, the nonanalytic terms are guaranteed to cancel, so only local operators are required.[b]

Comments:

- The factors of $\log M/m_e$ in the full theory become factors of $\log \mu/m_e$ in the matrix elements in the effective theory, along with corresponding factors of $\log M/\mu$ in the coefficient functions $c_2(\mu)$ and $c_3(\mu)$, where μ is the arbitrary scale introduced in $\overline{\text{MS}}$. Thus, μ is effectively a factorization scale, splitting the high energy physics which enters as parameters in the EFT from low energy physics, which enters through matrix elements. Furthermore, a finite log in the full theory (which is hard to calculate) has been replaced in the EFT by a logarithm related to the divergent part of the graph, which is easy to calculate.

[b] In non-relativistic EFT's (useful for studying charmonium and positronium, among other things), Lorentz invariance is broken, and nonlocal operators corresponding to instantaneous potentials are permitted[4].

- The renormalization prescription (in this case, dimensional regularization with $\overline{\mathrm{MS}}$) is part of the definition of the EFT. If we were to switch to another scheme, such as a cutoff, the matching conditions (26) would have to be modified accordingly.

- In the EFT, the M dependence of amplitudes is trivial - it only arises from the coefficient functions. Loop integrals never introduce additional M dependence. In particular, there can be no large logarithms $\log p/M$ in the EFT - as noted above, these become factors of $\log p/\mu$. This simplifies the renormalization group equations considerably.

- In practice, one never calculates matching conditions at an arbitrary external momentum and keeping the full p/m_e dependence, as in this example. Since the nonanalytic infrared dependence cancels in the matching conditions, the matching conditions may instead be calculated at any convenient kinematic point. In particular, once the leading factors of q have been pulled out of the integral, one can set $q = 0$ everywhere else. In general, amplitudes will be infrared divergent, but the divergences will cancel in the matching. By the same token, one can match in QCD at $\mu \gg \Lambda_{\mathrm{QCD}}$ using free quarks and gluons as external states, rather than hadrons. The fact that this description is not correct in the infrared cancels in the matching conditions.

1.3 Equations of Motion

When working in an EFT one typically has a large number of nonrenormalizable operators to include, particularly if working beyond leading order. A subtle point which is not always appreciated is that it is possible to use the equations of motion to simplify the operator basis[5,6]. For example, an operator like

$$\frac{1}{M}\bar{\psi}\left(i\not{D}-m\right)^2\psi \tag{27}$$

vanishes by the equations of motion, so can obviously be neglected at tree level. What is not so obvious is that it can be neglected in loop graphs as well, even though the operator will then act on virtual particles which aren't on shell. To show this, recall that according to the LSZ reduction formula, one is always free to perform a nonlinear field redefinition

$$\psi \to c\psi + [\text{anything}] \tag{28}$$

for any field ψ in the theory. Provided that the normalization of the new field is chosen so that it has a correctly normalized vacuum-to-one-particle matrix element, this redefinition will leave S-matrix elements unchanged. The advantage of this freedom is that *all operators which vanish by the equations of motion may be eliminated by an appropriately chosen field redefinition.*

In a renormalizable field theory, one doesn't usually perform nonlinear field redefinitions. For example, taking a free field theory with a scalar field of mass μ and performing the field redefinition

$$\varphi \to \varphi + \frac{1}{2} g \varphi^2 \tag{29}$$

gives the following complicated-looking Lagrangian,

$$\begin{aligned}\mathcal{L} = &\frac{1}{2} \left(\partial_\mu \varphi\right)^2 - \frac{1}{2}\mu^2 \varphi^2 + g \partial_\mu \varphi \partial^\mu \varphi \\ &+ \frac{1}{2} g^2 \partial_\mu \varphi \partial^\mu \varphi \varphi^2 - \frac{1}{2}\mu^2 g \varphi^3 - \frac{\mu^2}{8} g^2 \varphi^4 .\end{aligned} \tag{30}$$

This looks awful, but if you calculate with it you will find that all scattering amplitudes vanish, since it's just free field theory in disguise. However, in an EFT we already are including all possible operators, so such a field redefinition doesn't make things more complicated.

Consider now a theory with an operator which vanishes by the equations of motion,

$$\mathcal{L} = \bar{\psi}(i \not{D} - m)\psi + \frac{a}{\Lambda} \bar{\psi}(i \not{D} - m)^2 \psi . \tag{31}$$

Under the field redefinition

$$\psi' = \psi + \frac{a}{2\Lambda}(i \not{D} - m)\psi \tag{32}$$

this becomes

$$\mathcal{L} = \bar{\psi}'(i \not{D} - m)\psi' . \tag{33}$$

Since the two theories are guaranteed to give the same S-matrix elements, and the only difference is the term vanishing by the equations of motion, this term may clearly be neglected, as advertised. You should be able to convince yourself that a similar argument holds for any operator which vanishes by the equations of motion.

We are thus free to use the equations of motion in an EFT as *operator* statements. This changes off-shell amplitudes, but physical results like S-matrix elements are left unchanged. Thus, we don't need to include the operator

$$\bar{\psi} D^2 \psi \bar{\psi} \psi ; \tag{34}$$

since $D^2\psi = m^2\psi$, this is included in the operator $\bar\psi\psi\bar\psi\psi$. As another example, in the previous case of QED with a heavy scalar, the equation of motion for the photon field is

$$\partial_\mu F^{\mu\nu} = e\bar\psi\gamma^\nu\psi. \tag{35}$$

We can therefore trade the Darwin term for a four-fermi interaction

$$\bar\psi\partial_\mu F^{\mu\nu}\gamma_\nu\psi \to e\bar\psi\gamma_\mu\psi\bar\psi\gamma^\mu\psi. \tag{36}$$

As discussed in [6], the equations of motion are automatically implemented if one always uses *on-shell* external states when calculating matching conditions.

1.4 Scaling and the Renormalization Group

As they stand, neither Eq. (19) nor Eq. (23) is well-suited for perturbative calculations. The problem is the factor of $g^2 \log M^2/m_e^2$. If the ratio M^2/m_e^2 is large, such terms can invalidate perturbation theory.

In the EFT this large logarithm is split into two pieces via the arbitrary renormalization scale. The coefficient functions $c_2(\mu)$ and $c_3(\mu)$ contain a factor of $\log M^2/\mu^2$, while the matrix element (23) has a factor of $\log \mu^2/m_e^2$. Choosing $\mu \sim m_W$ makes perturbation theory well behaved for the matching conditions, but poorly behaved for calculations in the low-energy theory, whereas choosing $\mu \sim m_e$ reverses the situation. Thus, neither situation gives a good perturbative expansion. Instead, we would like to be able to choose $\mu \sim M$ to calculate the matching conditions reliably, and then lower μ to m_e so that we can calculate reliably in the EFT. This is exactly what the renormalization group equations allow us to do.

As a simpler example, let us forget QCD for a moment and return to the operator

$$O = \bar\mu\gamma^\mu(1-\gamma_5)\nu_\mu\bar u\gamma_\mu(1-\gamma_5)d, \tag{37}$$

which is relevant for charged-current neutrino hadron scattering. We write the effective weak Hamiltonian as a power series in the QED coupling $\alpha \equiv e^2/4\pi$:

$$\mathcal{H}_W = \frac{c(m_W,\mu)}{m_W^2}O(\mu) + \ldots, \tag{38}$$

where $O(\mu)$ denotes the operator renormalized at the scale μ (I have included the explicit m_W dependence in $c(\mu)$ to stress that the coefficient function depends on m_W, but not the external momentum).

Now consider higher-order QED corrections to the matrix element of $\mathcal{H}_W$. There will be corrections both to the matching conditions for $c(m_W,\mu)$ and to

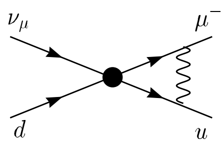

Figure 6: Graph contributing to the anomalous dimension of $O(\mu)$.

the matrix element. At one loop, the decay amplitude will be proportional to

$$c(m_W,\mu)\left[1-\gamma_0\frac{\alpha(\mu)}{4\pi}\log\frac{\mu}{\hat{m}}+\ldots\right] \tag{39}$$

where $\gamma_0=-4$ arises from the diagram in Fig. 6 (this is the only diagram with a large logarithm), the dots denote terms not enhanced by the large logarithm and $\hat{m}$ is a fictitious photon mass required to render the result infrared finite.

Now consider changing μ by an infinitesimal amount. Since the result is μ independent, it is easy to show that $c(m_W,\mu)$ will satisfy the differential equation

$$\mu\frac{d}{d\mu}c(m_W,\mu)=\gamma_0\frac{\alpha(\mu)}{4\pi}c(m_W,\mu) \tag{40}$$

where $d/d\mu$ is the total derivative, including the change in the coupling e with respect to μ:

$$\begin{aligned}\mu\frac{d}{d\mu}&=\mu\frac{\partial}{\partial\mu}+\beta(e)\frac{\partial}{\partial e}\\ \beta(e)&=\mu\frac{de}{d\mu}\equiv\beta_0\frac{e^3}{16\pi^2}+\ldots\end{aligned} \tag{41}$$

and $\beta_0=\frac{4}{3}$. The quantity $\gamma_0\frac{\alpha(\mu)}{4\pi}$ is known as the anomalous dimension of the operator O. The solution to the RGE is

$$c(m_W,\mu)=\left[\frac{\alpha(m_W)}{\alpha(\mu)}\right]^{-\frac{\gamma_0}{2\beta_0}}c(m_W,m_W). \tag{42}$$

Expanding this out gives

$$\begin{aligned}c(m_W,\mu)=\Bigg(1&-\frac{\alpha(\mu)}{4\pi}\gamma_0\log\frac{m_W}{\mu}+\frac{\alpha(\mu)^2}{32\pi^2}\gamma_0(\gamma_0+2\beta_0)\log^2\frac{m_W}{\mu}\\ &+\ldots\Bigg)c(m_W,m_W).\end{aligned} \tag{43}$$

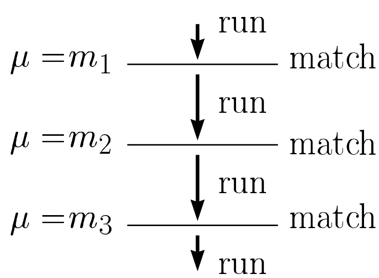

Figure 7: Scaling in a theory with multiple scales.

Since $c(m_W, m_W)$ has no large logs, we see that the complete series of leading logs has been summed by the RGE and put into the coefficient function $c(m_W, \mu)$.

The same approach can be used to sum leading logs in QCD, where the β function is

$$\beta(g) \equiv \mu \frac{dg}{d\mu} = -\left(11 - \frac{2}{3} N_f\right) \frac{g^3}{16\pi^2} \tag{44}$$

for N_f light flavours. In general, however, the RG equations will be complicated by operator mixing, and the equations will be a set of matrix equations for the various couplings $c_i(M, \mu)$ in the theory.

The combination of matching and the renormalization group gives a complete prescription for summing large logarithms in a theory with a number of distinct scales, as illustrated in Fig. 7. One begins at a high scale, where the full theory is known. One then runs down with the renormalization group equations until reaching a threshold (usually a particle mass) at $\mu = m_1$, and then matches onto a new EFT valid below $\mu = m_1$ (usually by integrating the particle out of the theory). The RG equations in the new EFT are then used to lower μ to the next threshold at $\mu = m_2$, and so on. Finally, at μ of the order of the typical external momentum in the problem, matrix elements are calculated. At each stage the renormalization scale μ is lowered, moving physics above that scale into the coefficients of the effective Lagrangian. Matching at tree level and solving the one-loop RGE resums leading logs of the form $\alpha_s^n \log^n p/m_i$; matching at one loop and running at two loops resums subleading logs $\alpha_s^{n+1} \log^n p/m_i$, and so forth.

Since QCD is strongly interacting at low energies, one typically cannot calculate matrix elements perturbatively. Nevertheless, this procedure correctly takes into account the calculable effects of gluons with large loop momenta,

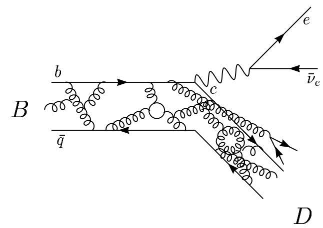

Figure 8: The wrong way to calculate $\bar{B} \to De\bar{\nu}_e$.

where the theory is still perturbative. Such effects are particularly important in theories where the EFT has additional symmetries at low energies which may be used to relate different matrix elements, since they contribute the leading symmetry-breaking effects due to virtual particles at high energies. We will see an example of this in the next section.

2 The $1/m_Q$ Expansion for Heavy Quarks

In this section we will put some of the ideas of the previous section to work in a phenomenologically important situation. Over the past few years, there has been a great deal of progress in understanding the dynamics of B and D mesons in the limit that the b and c quark masses are much greater than the typical QCD scale $\Lambda_{\rm QCD}$ (for a number of reviews, see [7,8].) As we will discuss, this limit is extremely nice because of the existence of additional symmetries, which allow model-independent, nonperturbative statements to be made. This situation, in which there is a large hierarchy of scales ($\Lambda_{\rm QCD} \ll m_Q$) is precisely the situation where EFT's simplify life considerably, and most of the progress in understanding this limit has been done in the context of the "heavy quark effective theory" (HQET).

2.1 Heavy Quark Symmetry

Consider, for the sake of definiteness, semileptonic $\bar{B} \to De\bar{\nu}_e$ decay. Based on our previous discussion, there are two ways to go about calculating this. The first, illustrated in Fig. 8, is to attempt to tackle the whole problem at once. This is clearly an impossible task in a strongly interacting theory. The diagram in Fig. 8 is not only a mess, but it contains all relevant scales of the problem mixed together: m_W (from the propagating W), m_b (the typical energy release in the decay), m_c (the invariant mass of the final state) and $\Lambda_{\rm QCD}$ (the typical

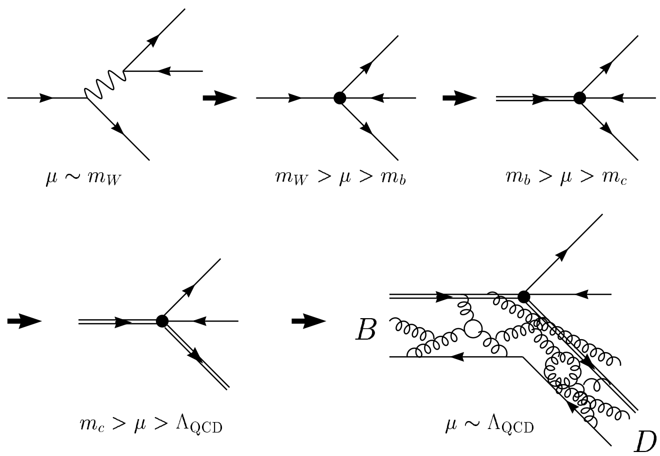

Figure 9: $\bar{B} \to De\bar{\nu}_e$ at a variety of scales μ.

energy of the strongly coupled gluons). This profusion of scales hides some essential simplifications which make the problem tractable.

It is more productive to think of the problem in terms of the effective theory relevant at each scale, as shown in Fig. 9. One begins at a high scale where the physics is simple and there are no large logarithms in the coefficients of the Lagrangian, and proceeds to lower the renormalization scale μ via the renormalization group equations, matching onto a new EFT at each particle threshold as discussed in the previous section.

Beginning at $\mu \sim m_W$, the process looks quite simple. A free b quark decays into a c quark and a W, which subsequently decays into the lepton pair. QCD is not strongly coupled at this scale, so processes with additional gluons are suppressed and may be perturbatively calculated. The appropriate EFT at this scale is just the usual $\mathrm{SU}(2)\times\mathrm{U}(1)$ theory of weak interactions. For $m_W > \mu > m_b$, the W can no longer be resolved, so the appropriate description of the decay is via the same four-fermi theory we discussed in the previous section. For $\mu < m_b$, things start to get interesting. In the previous section, when μ dropped below the threshold for a heavy particle it was integrated out of the effective theory. But here the situation is different. There is still a single heavy quark in the initial state, so it can't be integrated out of the theory. It is necessary to find the correct description of a massive, stable (with respect to the strong interactions) quark appropriate for momentum transfers much smaller than its mass.

Consider a quark of mass m_Q with momentum $p^\mu = m_Q v^\mu$, where v^μ is the four-velocity of the heavy quark. If it undergoes a soft momentum transfer $q \ll m_Q$, the change in v^μ is of order q/m_Q, and so vanishes as $m_Q \to \infty$. Thus, the four-velocity of the quark is a conserved quantity in the EFT: in the appropriate frame the quark is simply a static source of colour charge. So in the EFT below $\mu = m_Q$ the heavy quark should no longer be treated as a fully dynamical object, but rather as a source of colour charge moving with a velocity which is unchanged by its interactions. (We will discuss the simplifications this affords in a moment.) Such objects have actually been studied for quite some time in field theory, long before the introduction of HQET, and are known as Wilson lines. They are of interest because they are just classical colour sources.

Continuing to run μ down, at the scale $\mu = m_c$ one matches onto an EFT where the c quark is treated as a Wilson line. Finally, it's only at larger resolution, $\mu \sim \Lambda_{\rm QCD}$, that things get complicated and nonperturbative. Of course, this is still going to be a problem - writing things as an effective field theory gets us no closer to solving QCD. However, instead of QCD, one instead is faced with an EFT in which the dynamics of the b and c quarks are considerably simpler than in full QCD. In the appropriate frame, they just sit there, surrounded by their colour field. This looks like a trivial observation, but it immediately leads to several very powerful results.

- Flavour Symmetry: Once a quark is effectively a Wilson line, it doesn't matter how massive it is. A heavy quark can have a mass of 5 GeV or 5 kg; either way, it looks infinitely massive to a soft gluon. More practically, in the low energy theory a c quark looks exactly the same as a b quark. This statement is powerful because it means that there is a new *symmetry* in the effective theory - a flavour symmetry between heavy quarks. Whenever there is a symmetry in a theory, it allows nonperturbative statements to be made, just based on current conservation. This is the key to the power of the heavy quark expansion. Note that this is a rather peculiar flavour symmetry - unlike the usual flavour SU(3), it doesn't rely on the quark masses being approximately degenerate. All that is required is that the b and c masses both be much greater than $\Lambda_{\rm QCD}$. Since it exchanges particles with very different masses, the symmetry wasn't manifest in the QCD Lagrangian.

- Spin Symmetry: The magnetic moment of a particle scales like $1/m_Q$, so the interaction of a soft gluon with a heavy quark is spin-independent. Thus, the EFT should be invariant under rotations of the heavy quark spin.

- Pair Production: Since $p \ll 2m_Q$, there is no pair production in the effective theory, so particles and antiparticles decouple.

Thus, at leading order in $1/m_Q$, the EFT relevant for momenta much less than m_c is symmetric under the exchange of, for example, a spin up b quark with a spin down c quark: both states look identical to the light degrees of freedom. The new spin-flavour symmetry group for an EFT with N_f heavy flavours is

$$\mathrm{SU}(2N_f) \times \mathrm{SU}(2N_f) \tag{45}$$

where the first $\mathrm{SU}(2N_f)$ corresponds to particle transformations, and the second to antiparticles. (The theory isn't invariant under particle-antiparticle exchange, so the symmetry group isn't $\mathrm{SU}(4N_f)$.) Both of these symmetries will be exact only at leading order in the EFT; higher dimension operators scaling like $1/m_Q^n$ will in general break the symmetries. Thus, we should expect the symmetry breaking effects to be of order $\Lambda_{\mathrm{QCD}}/m_c \sim 25\%$ for the c quark, and $\sim 10\%$ for the b quark.

The simplification that has resulted from the EFT approach is twofold:

- The effects of virtual degrees of freedom at perturbative scales may be taken into account via perturbation theory. This is the usual matching and renormalization group running that was discussed in the last lecture, and sums leading logarithms of ratios of the various scales in the theory.

- The low energy degrees of freedom interact not with complicated, fully dynamical quarks, but rather with simple static colour charges. The additional $\mathrm{SU}(4) \times \mathrm{SU}(4)$ symmetry of this limit allows nonperturbative statements to be made.

Spectroscopy

The first place to look for evidence of this approximate symmetry is in the spectra of heavy hadrons. The momentum scale relevant to static hadronic properties is Λ_{QCD}, the typical momentum of the light degrees of freedom in the hadron. At this scale, a hadron containing a single heavy quark Q looks like a static source of colour charge interacting with light degrees of freedom (light quarks and gluons) which are in some horrible nonperturbative state (aptly referred to as the "brown muck" of QCD). Since we can't solve QCD, the properties of this state aren't analytically calculable. However, the symmetry (45) may be used to relate the light degrees of freedom in different hadrons, since the state of the light degrees of freedom is independent of the spin and flavour of Q. Thus, states which differ only in the relative spin orientation

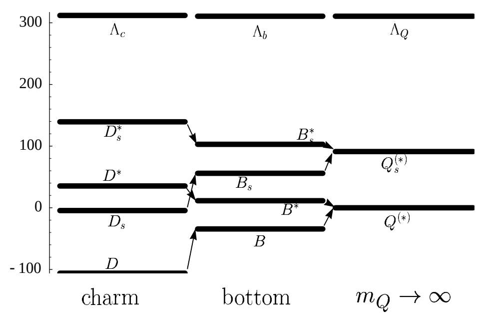

Figure 10: Spectra of heavy hadrons. The excitation energies of the states are relative to the spin-averaged mass of the lowest lying doublet, $\bar{m}_Q \equiv (m_Q + 3m_{Q^*})/4$.

of the heavy quark and light degrees of freedom should be degenerate, up to corrections suppressed by powers of $\alpha_s(m_Q)$ and $\Lambda_{\rm QCD}/m_Q$. Pairs of such states include the D and D^* pseudoscalar and vector states, the $D_1(2420)$ and $D_2^*(2460)$ pseudovector and tensor states, and the corresponding states in the b system.

The heavy quark flavour symmetry ensures that the excitation energy (that is, the mass of the state minus the heavy quark mass) of a given state is independent of the flavour of the quark Q. In Fig. 10 the masses of some of the low-lying mesons in the b and c systems are plotted, along with the $m_Q \to \infty$ limit. It is clear that these expectations hold to a reasonable approximation.

Furthermore, as expected, the hyperfine pseudoscalar-vector splittings are smaller for the b system than for the c system. This can be made more quantitative. Since the spitting is a magnetic moment effect and so scales like $1/m_Q$, we expect

$$\frac{m_{B^*} - m_B}{m_{D^*} - m_D} = \frac{m_c}{m_b} \simeq \frac{m_D}{m_B}. \tag{46}$$

Putting in the experimental numbers, this gives

$$0.32 \simeq 0.35 \tag{47}$$

and so the hyperfine splitting indeed scales like $1/m_Q$.

Semileptonic Decays

Semileptonic $b \to c$ decays are probably the most important application of the approximate symmetries that arise in HQET. Even though we haven't solved any of the problems of hadronization, symmetry arguments again greatly restrict the form of the result.

Consider the semileptonic decays $\bar{B} \to De\bar{\nu}_e$ and $\bar{B} \to D^* e\bar{\nu}_e$. On the basis of Lorentz invariance, the relevant amplitudes may be decomposed in terms of form factors,

$$\begin{aligned}
\langle D(v')|\bar{c}\gamma^\mu b|B(v)\rangle &= h_+(w)(v_\mu + v'_\mu) + h_-(w)(v_\mu - v'_\mu) \\
\langle D(v')|\bar{c}\gamma^\mu \gamma_5 b|B(v)\rangle &= 0 \\
\langle D^*(v',\epsilon)|\bar{c}\gamma^\mu b|B(v)\rangle &= ih_V(w)\epsilon_{\mu\nu\alpha\beta}\epsilon^*_\nu v'_\alpha v_\beta \\
\langle D^*(v',\epsilon)|\bar{c}\gamma^\mu \gamma_5 b|B(v)\rangle &= h_{A_1}(w)(w+1)\epsilon^{*\mu} \\
&\quad -\epsilon^* \cdot v\, [h_{A_2}(w)v^\mu + h_{A_3}(v)v'^\mu]
\end{aligned} \tag{48}$$

where v and v' are the four-velocities of the hadrons, $w = v \cdot v'$, and ϵ is the polarization of the D^*. The form factors h_i depend on the details of hadronization. They are not calculable in perturbation theory, and are *a priori* independent functions, each of which must be measured experimentally. However, in the heavy quark limit they are all related by symmetry. To see this physically, consider the semileptonic decay in the low energy theory. A b quark moving with constant velocity v^μ surrounded by brown muck in its ground state suddenly decays to another colour source, this time moving with some other constant velocity v'^μ, due to the large energy release in the decay. The only property of the charmed final state which is relevant to the decay is its relative velocity to the initial b quark - in particular, its spin and flavour are completely irrelevant, as our symmetries assured us. Thus, the various form factors in Eq. (48) must all be proportional to the overlap between the light degrees of freedom in a $\bar{B}$ meson moving with a velocity v and the light degrees of freedom in a D meson moving with velocity v'. Somewhat schematically, we write the overlap between the light degrees of freedom as a function of w,

$$\langle \text{light degrees of freedom}, v|\text{light degrees of freedom}, v'\rangle = \xi(w). \tag{49}$$

Classifying the states in terms of their transformation properties under the spin-flavour symmetry, it can be shown with a minimal amount of group theory[9,10] that the form factors satisfy

$$\begin{aligned}
&h_+(w) = h_V(w) = h_{A_1}(w) = h_{A_3}(w) = \xi(w), \\
&h_-(w) = h_{A_2}(w) = 0
\end{aligned} \tag{50}$$

where the universal function $\xi(w)$ is known as the *Isgur-Wise* function. Heavy quark symmetry has reduced seven unknown and *a priori* independent form factors to a single unknown function $\xi(w)$. Similar arguments may be made for B decays to other exclusive charmed final states, although since the light degrees of freedom are in different states, there will be a different universal function for each spin multiplet.

For semileptonic $\bar{B} \to D^{(*)}$ decays there is even more nonperturbative information available. Because of the spin and flavour symmetries, the light degrees of freedom are in the *same* state (except for the boost) in both the initial and final hadrons, regardless of whether the final hadron is a D or a D^*. Therefore, at the kinematic point where the velocities of the initial and final quarks are the same, from the point of view of the light degrees of freedom, then, absolutely nothing has happened! The overlap between the initial and final states is unity. (This is the point where the invariant mass q^2 of the lepton pair is a maximum). Thus, we have the remarkable property that the Isgur-Wise function is known at one point:

$$\xi(1) = 1. \tag{51}$$

(Note that a corresponding statement for $\bar{B}$ decays to other charmed states is not true. Rather, the relevant matrix element at zero recoil to states other than D or D^* vanishes in the heavy quark limit, since the light degrees of freedom in the states are orthogonal). The phenomenological significance of this relation should be clear - at this one kinematic point, there are no hadronic uncertainties in the calculation of the decay rate at leading order in $1/m_{b,c}$ and α_s. This gives one the opportunity to measure the CKM matrix element V_{bc} in a model-independent way. Since this discovery in the original Isgur-Wise papers[9], much of the effort in HQET has concerned calculating the corrections to this result. In order to do this, in the next section we will explicitly construct the EFT for heavy quarks.

2.2 Heavy Quark Effective Theory

Having discussed some of the physical consequences of the enhanced symmetries of QCD with heavy quarks, we now turn to the construction of the appropriate EFT. This will allow us to calculate (or at least parameterize) the corrections to the heavy quark limit in a systematic way.

Recall that we are expanding the theory in powers of the typical momentum of a light degree of freedom over the mass of the heavy quark. This may be made more quantitative by noting that if a heavy hadron is moving with four-velocity v^μ, the bulk of its four-momentum is carried by the heavy quark. The quark momentum may then be separated into a “large” piece (scaling like

m_Q) and a "small" piece (scaling like $\Lambda_{\rm QCD}$):

$$p_Q^\mu = m_Q v^\mu + k^\mu. \tag{52}$$

k^μ is often referred to as the "residual" momentum of the heavy quark, and its size is set by $\Lambda_{\rm QCD}$. Now, in the limit $m_Q \to \infty$, interactions with the light degrees of freedom cannot change v^μ, but only k^μ. v^μ is a therefore a conserved quantity in the EFT, and may be used to label heavy quark states. To construct the kinetic term for this object, we expand the heavy quark propagator in powers of k^μ/m_Q:

$$\frac{i(\not{p}+m_Q)}{p^2-m_Q^2} = \frac{im_Q(\not{v}+1+\not{k}/m_Q)}{2m_Q v\cdot k + k^2} = \frac{i}{v\cdot k}\left[\frac{1+\not{v}}{2}\right] + O(1/m_Q) \tag{53}$$

$\frac{1+\not{v}}{2}$ is a projection operator, projecting out the "large" Dirac components of the heavy quark spinor. Thus, the degrees of freedom in HQET are two-component spinors h_v satisfying

$$\left(\frac{1+\not{v}}{2}\right) h_v = h_v, \quad \left(\frac{1-\not{v}}{2}\right) h_v = 0. \tag{54}$$

with Lagrangian

$$\mathcal{L} = \bar{h}_v iD\cdot v h_v + O(1/m_Q). \tag{55}$$

The large component of the heavy quark momentum has also been removed from h_v by pulling out a phase

$$h_v \sim e^{imv\cdot x}\psi_Q \tag{56}$$

(where ψ_Q is the field in the full theory[c]) and so there are no large dynamical momenta in the EFT. Because of this phase redefinition, the free fields satisfy

$$\partial_\mu h_v = -ik_\mu h_v \tag{57}$$

so derivatives just pull down factors of the residual momentum. Thus, the derivative expansion is an expansion in powers of k^μ/m_Q, as required. Note also that sandwiching a γ^μ between projection operators just gives v^μ:

$$\left(\frac{1+\not{v}}{2}\right)\gamma^\mu\left(\frac{1+\not{v}}{2}\right) = \left(\frac{1+\not{v}}{2}\right) v^\mu \tag{58}$$

and so the heavy quark-gluon coupling has Feynman rule $ig_s v^\mu T^a$. This this is exactly what one gets from the covariant derivative in (55).

[c]One shouldn't take this relation too literally. There is no simple relation between the full heavy quark field and the field in the EFT, as their high-momentum modes are completely different.

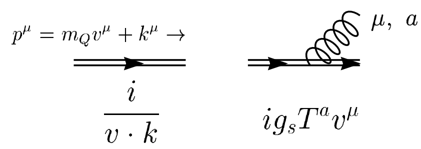

Figure 11: Feynman rules for HQET at leading order in $1/m_Q$.

For a theory with heavy b and c quarks, the leading order Lagrangian is (in obvious notation for the two flavours)

$$\mathcal{L} = \bar{b}_v iD \cdot v b_v + \bar{c}_v iD \cdot v c_v + O(1/m_b,\ 1/m_c). \tag{59}$$

$\mathcal{L}$ is invariant under flavour rotations between b_v and c_v (note that the flavour symmetry interchanges quarks of equal velocities, not momenta) as well as spin rotations (since there is no γ structure in $\mathcal{L}$ at this order). Thus, the $\mathrm{SU}(4) \times \mathrm{SU}(4)$ symmetry we argued for on physical grounds is manifest in the effective Lagrangian.

Furthermore, in the EFT we can easily understand the normalization condition $\xi(1) = 1$ in terms of the symmetry. The currents

$$\bar{c}_v \Gamma b_v, \quad \bar{b}_v \Gamma c_v, \quad \bar{b}_v \Gamma b_v, \quad \bar{c}_v \Gamma c_v \tag{60}$$

(where $\Gamma = \gamma^\mu$ or $\gamma^\mu \gamma_5$) are all conserved currents corresponding to the spin-flavour symmetry. Thus, their matrix elements are related to a conserved charge, and so are fixed. Explicitly,

$$\langle B_v | \bar{b}_v \gamma^\mu b_v | B_v \rangle \tag{61}$$

is the b-quark current, which just counts the number of b's in a B meson, and so its matrix element is fixed to 1 (with appropriate normalization). But we can do a flavour rotation, followed by a spin rotation, to relate this to any of the matrix elements of interest:

$$\langle B_v | \bar{b}_v \gamma^\mu b_v | B_V \rangle \sim \langle D_v | \bar{c}_v \Gamma b_v | B_V \rangle \sim \langle D_v^* | \bar{c}_v \Gamma b_v | B_V \rangle \tag{62}$$

for any Dirac matrix Γ. Thus, all weak matrix elements in $\bar{B} \to D(D^*) e \bar{\nu}_e$ are fixed at zero recoil ($v = v'$) by the symmetries of the effective theory. Away from zero recoil, we can still use spin-flavour symmetries to relate all of the form factors to a single form factor, $\xi(v \cdot v')$. For example, under a spin rotation

$$\langle D_{v'} | \bar{c}_{v'} \gamma^\mu b_v | B_V \rangle \sim \langle D_{v'}^*(\epsilon) | \bar{c}_{v'} \gamma^\mu b_v | B_V \rangle \tag{63}$$

which relates $h_+(v \cdot v')$ and $h_V(v \cdot v')$ as defined earlier,

$$h_+(v \cdot v') = h_V(v \cdot v') = \xi(v \cdot v') \tag{64}$$

up to $O(1/m_{b,c})$ corrections. Similar (and more careful, using the transformation properties of the states and currents under the $\mathrm{SU}(4) \times \mathrm{SU}(4)$ symmetry) arguments give the compete set of relations in Eq. (50).

Example: Decay Constants

The decay constant of a heavy pseudoscalar meson M, f_M, containing a heavy quark Q and a light quark q, is defined as

$$\langle M|\bar{Q}\gamma^\mu\gamma_5 q|0\rangle \equiv i p_Q^\mu f_M. \tag{65}$$

This matrix element cannot be calculated in perturbation theory; however, heavy quark flavour symmetry relates the constants f_B and f_D. The only subtlety in this relation is the normalization of the states, since in the standard relativistic normalization the states are proportional to $\sqrt{m_Q}$. Pulling out this normalization, and noting that $p_Q \sim m_Q$, the prediction of heavy quark symmetry is

$$\frac{f_D}{f_B} = \sqrt{\frac{m_B}{m_D}} + O\left(\alpha_s, \frac{\Lambda_{\mathrm{QCD}}}{m_{b,c}}\right). \tag{66}$$

Now let us consider radiative corrections to this result. At one loop in full QCD, the matrix element of the heavy quark operator would contain large logarithms, of order $\alpha_s \log m_b/\lambda$, where λ is the typical “soft” momentum scale in the problem. The leading logs may be summed in the EFT by matching at tree level and running at one loop with the renormalization group equations[d].

Above $\mu = m_b$ the current $\bar{b}\gamma^\mu\gamma_5 q$ is partially conserved, and so has zero anomalous dimension. There are thus no logarithms of m_W/m_b to sum. At the scale $\mu = m_b$, we match onto a theory with a single heavy quark. The original current then becomes, in the effective theory,

$$\bar{b}\gamma^\mu\gamma_5 q \to c(m_b)\bar{b}_v\gamma^\mu\gamma_5 q + \dots \tag{67}$$

where the dots denote higher dimension operators (such as $\bar{b}_v \not{D}\gamma^\mu\gamma_5 q$) which are suppressed by powers of $1/m_Q$. At tree level, the matching is trivial: $c(m_b) = 1$.

[d]See also Ref.[11], where the running was calculated in the full theory, by summing the finite logs of m_b/λ. The calculation is much simpler in the EFT.

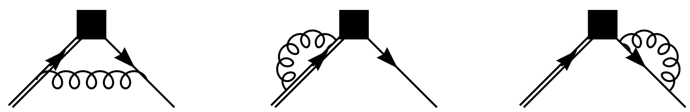

Figure 12: Diagrams contributing to the one-loop anomalous dimension of $\bar{b}_v\gamma^\mu\gamma_5 q$.

Now we run the operator down to the next interesting momentum scale, m_c. The anomalous dimension of the heavy-light current is calculated from the graphs in Fig. 12,

$$\gamma = 4\frac{\alpha_s}{\pi}. \tag{68}$$

At $\mu = m_c$, the only change in the theory is that the c quark is integrated out, so the β function changes. Below $\mu = m_c$, the operator continues to run, this time in a theory with three light flavours. The result is

$$c(\mu) = \left(\frac{\alpha_s(m_b)}{\alpha_s(m_c)}\right)^{6/25}\left(\frac{\alpha_s(m_c)}{\alpha_s(\mu)}\right)^{6/27} \tag{69}$$

The effects of the gluons with momenta in the region μ and m_b have now been explicitly taken into account analytically - the first factor comes from running between m_b and m_c, and the second from m_c to μ.

Furthermore, this also gives the leading flavour symmetry breaking corrections to the ratio f_B/f_D. Since the operator $\bar{c}\gamma^\mu\gamma_5 q$ doesn't start running until $\mu = m_c$, we obtain the ratio

$$\frac{f_B}{f_D} = \sqrt{\frac{m_D}{m_B}}\left(\frac{\alpha_s(m_b)}{\alpha_s(m_c)}\right)^{6/25} + O(1/m_{c,b},\, \alpha_s^2\log(m_b/m_c)). \tag{70}$$

The running is about a 10% effect. Physically, the symmetry breaking effect arises because virtual gluons between $\mu = m_c$ and $\mu = m_b$ can distinguish c from b quarks.

Higher Orders in $1/m_Q$

In addition to symmetry-breaking corrections due to virtual gluons, there are corrections to our results suppressed by powers of $\Lambda_{\rm QCD}/m_c$ and $\Lambda_{\rm QCD}/m_b$ from nonrenormalizable operators in the EFT. Since $\Lambda_{\rm QCD}/m_c \sim 0.2$, such effects are crucial if HQET is to be useful for precision physics. But as in any EFT, we can work to arbitrary precision in the ratio of scales, with the tradeoff being the introduction of more and more unknown matrix elements. Fortunately, as we will see, it is possible to make statements which are correct beyond leading order in $1/m_c$, giving us real precision calculations.

It is straightforward to extend the Lagrangian to higher orders in $1/m_Q$. There are only two dimension five operators (corresponding to $O(1/m_Q)$) operators we can write down in the EFT,

$$\mathcal{L}_1 = \frac{1}{2m_Q}\left(c_1(\mu)\bar{h}_v(iD)^2 h_v + c_2(\mu)\bar{h}_v\sigma^{\alpha\beta}G_{\alpha\beta}h_v\right). \tag{71}$$

There is an additional dimension five operator, $\bar{h}_v(iD\cdot v)h_v$, but this vanishes by the equation of motion,

$$D\cdot v h_v = 0 + O(1/m_Q) \tag{72}$$

and so may be neglected. By considering their action on free fields, we see that O_1 is the kinetic energy term $(p^2/2m_Q)$, whereas O_2 is the chromomagnetic moment operator that we saw in the previous section. Note that O_1 violates heavy flavour symmetry, while O_2 violates both spin and flavour symmetry. This justifies our previous assertion that the pseudoscalar-vector mass splittings scale like $1/m_Q$.

It is easiest to calculate c_1 and c_2 by considering the gluon-quark-quark vertex in QCD. This can be expanded in powers of $1/m_Q$ via the Gordon decomposition:

$$\begin{aligned}\bar{u}(p')\gamma^\mu u(p) &= \frac{(p^\mu + p'^\mu)}{2m_Q}\bar{u}(p')u(p) + i\frac{(p^\nu - p'^\nu)}{2m}\bar{u}(p')\sigma_{\mu\nu}u(p)\\ &= v^\mu\bar{u}(p')u(p) + \frac{1}{2m_Q}(k^\mu + k'^\mu)\bar{u}(p')u(p) + i\frac{p_\nu - p'_\nu}{2m_Q}\bar{u}(p')\sigma^{\mu\nu}u(p)\\ &\quad +\ldots.\end{aligned} \tag{73}$$

This gives the tree-level matching conditions

$$c_1(m_Q) = 1, \quad c_2(m_Q) = \frac{g}{2}. \tag{74}$$

Given the Lagrangian to $O(1/m_Q)$, to calculate the matrix element of an operator O to this order one must include matrix elements of the time-ordered products of O with the $1/m_Q$ operators in $\mathcal{L}$. Such matrix elements cannot be calculated in a strongly interacting theory, and so must be parameterized (again, in an $SU(2N_f)\times SU(2N_f)$ invariant way), reducing the predictive power of the EFT. In addition, the operator O (for example, a weak current) must be matched to $O(1/m_Q)$, introducing more unknown matrix elements. We will see this explicitly in the next section.

One can, in principle, continue this indefinitely. However, even at $O(1/m_Q^2)$ the EFT is rather complicated[12]. In addition to the Darwin and spin-orbit terms,

$$O_D = \frac{g}{8m^2}\bar{h}\,(D_\mu G^{\mu\nu})\,v_\nu h, \qquad O_S = i\frac{g}{8m^2}\bar{h}\sigma_{\mu\nu}\{D^\mu, G^{\rho\nu}\}v_\rho, \tag{75}$$

there are eight (!) four-fermion operators, one "penguin" operator and two triple-gluon operators. The equations of motion simplify this list somewhat, but there are still clearly too many operators to do much that's useful, beyond estimating the size of the effects.

Application: Extracting $|V_{bc}|$ from $\bar{B} \to D^{(*)} e\bar{\nu}_e$

Let us now use the EFT machinery to calculate the corrections to the prediction of the absolute normalization of the form factors for $\bar{B} \to D^* e\bar{\nu}_e$ at zero recoil. Quite generally, we can write the differential decay rate

$$\begin{aligned}\frac{d\Gamma}{dw}(\bar{B} \to D^{(*)} e\bar{\nu}_e) = {}& \frac{G_F^2}{48\pi^3}|V_{bc}|^2(m_B - m_{D^*})^2 m_{D^*}^3 (w+1)^3\sqrt{w^2-1} \\ & \times\left[1 + \frac{4w}{w+1}\frac{m_B^2 - 2wm_Bm_{D^*} + m_{D^*}^2}{(m_B - m_{D^*})^2}\right]|F(w)|^2\end{aligned} \tag{76}$$

where

$$F(w) = \xi(w) + O(\alpha_s) + O(1/m_{c,b}) \tag{77}$$

and the complicated w dependence just comes from the phase space integrals.

Radiative Corrections: These arise from perturbative corrections to matching conditions, as well as running between $\mu = m_b$ and $\mu = m_c$, and so are completely calculable. In fact, the one-loop running calculation is the same one we encountered for the ratio of decay constants. The sequence of matchings is that shown in Fig. 9,

$$\begin{aligned}\bar{c}\gamma^\mu(1-\gamma_5)b &\overset{\mu=m_b}{\longrightarrow} c(m_b)\bar{c}\gamma^\mu(1-\gamma_5)b_v \\ &\overset{\text{run}}{\longrightarrow} c(m_c)\bar{c}\gamma^\mu(1-\gamma_5)b_v \overset{\mu=m_c}{\longrightarrow} c'(m_c)\bar{c}_v\gamma^\mu(1-\gamma_5)b_v.\end{aligned} \tag{78}$$

Since the current is conserved for $v = v'$, its anomalous dimension vanishes below $\mu = m_c$. From our previous results, then, we get the leading perturbative correction to our results,

$$\bar{c}\gamma^\mu(1-\gamma_5)b \to \left(\frac{\alpha_s(m_b)}{\alpha_s(m_c)}\right)^{6/25} \bar{c}_v\gamma^\mu(1-\gamma_5)b_v \tag{79}$$

and so

$$F(1) = \left(\frac{\alpha_s(m_b)}{\alpha_s(m_c)}\right)^{6/25} + O(\alpha_s^{n+1}\log^n m_c/m_b) + O(1/m_{b,c}). \tag{80}$$

At next order, the current must be matched to $O(\alpha_s)$ accuracy, and the running performed to two loops (resumming all terms of order $\alpha_s^{n+1}\log^n m_c/m_b$). This calculation has been done[13], and is included below.

Power Corrections: Just as the Lagrangian has corrections corresponding to higher dimension operators, so the weak current matches in the EFT onto additional operators. For general v and v', the only operator at $O(1/m_c)$ is

$$\bar{c}_{v'} i\overleftarrow{\not{D}}\gamma^\mu(1-\gamma_5)b_v \tag{81}$$

and so at $O(1/m_c)$ we need the matrix element

$$\langle D^{(*)}(v')|\bar{c}_{v'} i\overleftarrow{\not{D}}\gamma^\mu(1-\gamma_5)b_v|B(v)\rangle \tag{82}$$

There are three possible form factors required to describe the matrix elements of this operator in HQET. Furthermore, additional $1/m_Q$ corrections arise due to T-products of $1/m_Q$ operators in the effective Lagrangian and the leading order current,

$$\langle D^{(*)}(v')|T(\bar{c}_{v'}\gamma^\mu b_v, \mathcal{L}_1)|B(v)\rangle. \tag{83}$$

However, at zero recoil ($v = v'$) things simplify dramatically: both of these corrections *vanish*, although each for a different reason[14].

- The T-product in (83) vanishes by the Ademollo-Gatto theorem when $v = v'$. The theorem states that the symmetry breaking corrections to matrix elements of an approximate symmetry current are second order in the breaking terms in the Lagrangian.

- The matrix element (82) can be related, when $v = v'$, to the matrix element of the operators $\bar{c}_v\overleftarrow{D}\cdot v b_v$, which vanishes by the equations of motion.

Thus, we have the nice result that the leading nonperturbative effects to the absolute normalization of the Isgur-Wise function at zero recoil are of order $\Lambda_{\rm QCD}/m_c^2$, not $\Lambda_{\rm QCD}/m_c$. This immediately raises this prediction to a reasonable level of precision, since corrections of order $\Lambda_{\rm QCD}^2/m_c^2$ are expected to be of order 5%.

Putting everything together, we find

$$\begin{aligned} F(1) &= 1 + \eta_A^{(1)} \frac{\alpha_s(m_b)}{\pi} + \eta_A^{(2)} \left(\frac{\alpha_s(m_b)}{\pi} \right)^2 + \delta_{1/m^2} + O\left(\alpha_s^3, 1/m_{b,c}^3\right) \\ &= 0.960 \pm 0.007 + \delta_{1/m^2} + O\left(\alpha_s^3, 1/m_{b,c}^3\right) \end{aligned} \tag{84}$$

where δ_{1/m^2} refers to the incalculable $O(1/m_{c,b}^2)$ corrections. There have been a number of attempts to estimate the size of these corrections from quark models, sum rules and other methods. Combining the results in the literature gives the rough estimate

$$\delta_{1/m^2} = -5.5 \pm 3\% \tag{85}$$

which gives the result

$$F(1) = 0.91 \pm 0.04. \tag{86}$$

Combining this with the measured value of $d\Gamma/dw$ at the endpoint (see Ref. [15] for a compilation of the experimental results) gives the value

$$|V_{bc}| = 0.0376 \pm 0.0015_{\rm expt.} \pm 0.0012_{\rm theory}. \tag{87}$$

2.3 Other Applications of the $1/m_Q$ Expansion

The heavy quark expansion has becoming the starting point for much of B phenomenology today. Another application of particular interest is the case of *inclusive* decays, such as

$$\bar{B} \to \sum_{X_c} X_c e \bar{\nu}_e \tag{88}$$

where all charmed final hadronic states X_c are summed over. In this case, the heavy quark expansion may be used to show that for sufficiently inclusive quantities (such as the total decay width, the electron spectrum in B decays, or moments of the invariant mass spectrum in B decays) the free quark result calculated in perturbation theory

$$b \to c \bar{\nu}_e + \text{anything} \tag{89}$$

is the leading term in a $1/m_Q$ expansion. Just as in HQET, the subleading effects may be systematically taken into account through higher dimension operators. Such an approach has had many applications, particularly in providing an alternative method of extracting V_{bc} (which agrees well with the result presented here), as well as for studying other decays such as semileptonic $b \to u$ and the rare decays $B \to X_s \gamma$ and $B \to X_s e^+ e^-$ (for reviews, see [8]).

3 EFT's Without Matching: Chiral Perturbation Theory

In the examples of effective field theories we have seen thus far, the theories were weakly coupled at the matching scale, and so the matching conditions were perturbatively calculable. Thus, the coefficients of all operators in the EFT were calculable, at least in principle.

However, if the underlying theory is nonperturbative at the matching scale, the coefficients in the EFT cannot be calculated perturbatively. The classic example of this situation is chiral perturbation theory, the EFT describing the interactions of the light pseudoscalar mesons $\pi,\ K, \eta$ at low momenta. Despite the fact that the coefficients of the EFT are not perturbatively calculable, the symmetries and power counting may be used to obtain a remarkable amount of predictive power. In this section, we will develop the formalism of chiral perturbation theory, and discuss some of its phenomenological applications. For reviews of the subject, see [16,17,18].

3.1 Chiral Symmetry Breaking

Why is the pion so light? By light, we mean light compared to the typical masses one encounters in QCD. The pion mass is a nonperturbative effect, and the typical mass scale associated with nonperturbative effects in QCD is ~ 1 GeV, roughly the mass of the proton. The pion, in contrast, weighs in at a scant ~ 140 MeV.

Recall that it is very unnatural to have very light particles in a theory, unless there is an approximate symmetry which makes them massless. For the pions (and the rest of the octet of light pseudoscalars, the kaons and the eta) this is indeed the reason - these particles are the (pseudo)-Goldstone bosons (PGB's) of the approximate $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ chiral symmetry of QCD, which is spontaneously broken by QCD dynamics to the diagonal subgroup, $\mathrm{SU}(3)_{L+R}$. Because of this, the interactions of the light octet are strongly constrained by the symmetries of the theory. To see how this comes about in an EFT, let's first consider a simpler, familiar case.

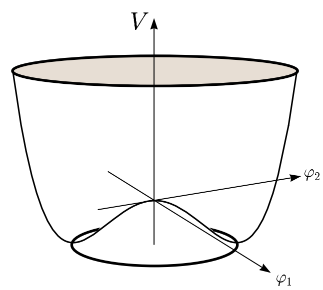

Figure 13: The potential in Eq. (90). The vacuum lies on any point on the minimum, and spontaneously breaks the $O(2)$ symmetry.

Example: Spontaneously broken $O(N)$

Consider a simple theory of a two real scalar fields,

$$\mathcal{L} = \sum_{i=1}^{2} \frac{1}{2}\partial_\mu \varphi_i \partial^\mu \varphi_i - \frac{\lambda}{4}\left(\varphi_1^2 + \varphi_2^2 - v^2\right)^2 . \tag{90}$$

The Lagrangian has an $O(2)$ symmetry. However, the minimum of the potential lies along the curve $\varphi_1 + i\varphi_2 = ve^{i\theta}$, as shown in Fig. 13. All values of θ are equivalent, however for any value of θ the the $O(2)$ symmetry is spontaneously broken. Goldstone's theorem tells us that there is a massless excitation, a Goldstone boson, corresponding to excitations of the field along the minimum of the potential. Choosing the vacuum to lie in the φ_1 direction,

$$\langle \varphi_1 \rangle = v, \ \langle \varphi_2 \rangle = 0 \tag{91}$$

and defining the shifted fields

$$\tilde{\varphi}_1 = \varphi_1 - v, \ \tilde{\varphi}_2 = \varphi_2 \tag{92}$$

the Lagrangian reads

$$\begin{aligned} \mathcal{L} = &\frac{1}{2}\partial_\mu \tilde{\varphi}_1 \partial^\mu \tilde{\varphi}_1 + \frac{1}{2}\partial_\mu \tilde{\varphi}_2 \partial^\mu \tilde{\varphi}_2 - \frac{\lambda v^2}{2}\tilde{\varphi}_1^2 \\ &- \frac{\lambda v}{2\sqrt{2}}\tilde{\varphi}_1 \tilde{\varphi}_2^2 - \frac{\lambda v}{2\sqrt{2}}\tilde{\varphi}_1^3 - \frac{\lambda}{16}(\tilde{\varphi}_1^2 + \tilde{\varphi}_2^2)^2 \end{aligned} \tag{93}$$

In accordance with Goldstone's theorem, the $\tilde{\varphi}_2$ field is a massless Goldstone boson (GB), while the $\tilde{\varphi}_1$ field has a mass $m_1 = \sqrt{\lambda}v$. However, this choice of fields obscures some important physics. Instead, let us combine the two real fields into a single complex field $\phi = \varphi_1 + i\varphi_2$, and write ϕ in terms of radial and angular fields $\rho(x)$ and $\theta(x)$,

$$\phi = (v + \rho(x))\, e^{i\theta(x)/v}. \tag{94}$$

The broken $O(2)$ symmetry is equivalently a $U(1)$ transformation of ϕ,

$$\phi \to e^{i\lambda}\phi, \tag{95}$$

and in terms of the angular variables, the Lagrangian is

$$\begin{aligned}\mathcal{L} = & \frac{1}{2}\partial_\mu\rho\partial^\mu\rho - \frac{\lambda v^2}{2}\rho^2 + \frac{1}{2}\partial_\mu\theta\partial^\mu\theta \\ & + \frac{1}{\sqrt{2}v}\rho(\partial_\mu\theta\partial^\mu\theta) + \frac{1}{4v^2}\rho^2\partial_\mu\theta\partial^\mu\theta - \frac{1}{2\sqrt{2}}\lambda v\rho^3 - \frac{\lambda}{16}\rho^4.\end{aligned} \tag{96}$$

In this form, we see that the Goldstone boson θ is *derivatively* coupled. Thus, its interactions are proportional to its momentum, and as $p \to 0$ it becomes a free field. This is clear geometrically, since the potential is only a function of the radial direction and not the angle, so there can be no non-derivative terms containing $\theta(x)$ in the Lagrangian. The derivative couplings of Goldstone bosons is another general consequence of the spontaneous symmetry breaking of a global symmetry, and will be very important in our discussion of pion interactions.

The mass of the heavy mode (the $\tilde{\varphi}_1$ or the ρ) in this theory is proportional to v. Thus, if we are now interested only in the low-momentum ($p^\mu \ll v$) interactions of the Goldstone bosons, we should integrate out the massive degree of freedom and construct an EFT of only the Goldstone boson. In terms of the angular variables, the resulting EFT will clearly only contain derivative couplings. In terms of the $\tilde{\varphi}_2$ field this is not obvious (since the $\tilde{\varphi}_2$ direction is only tangent to the vacuum manifold, and so the potential is not flat in that direction). However, if you calculate the matching conditions, you will find that the contribution from integrating out φ_1 exactly cancels the non-derivative couplings in $\mathcal{L}$, again leaving a derivatively coupled theory.

Equivalently, since at low energies the only allowed excitations lie on the vacuum manifold $|\phi|^2 = v^2$, let us define

$$\Xi = e^{i\theta(x)/v}, \quad \langle \Xi \rangle = 1. \tag{97}$$

Ξ is just as good a field as $\tilde{\varphi}_2$ or θ to describe the dynamics of ϕ when it is constrained to lie on the vacuum manifold, as it is in the EFT below v. In this representation, the field Ξ transforms linearly under the broken $U(1)$,

$$\Xi \to e^{i\lambda}\Xi \tag{98}$$

whereas the θ field shifts by a constant,

$$\theta \to \theta + v\lambda. \tag{99}$$

The low energy EFT

$$\mathcal{L}_{\rm eft} = \frac{v^2}{2}\left[\partial_\mu \Xi^\dagger \partial^\mu \Xi + \frac{a_1}{v^2}(\partial_\mu \Xi^\dagger \partial^\mu \Xi)^2 + \ldots\right] \tag{100}$$

is invariant under the spontaneously broken $U(1)$

$$\Xi \to e^{i\lambda}\Xi \tag{101}$$

and the a_i's are calculable via the usual matching procedure. The EFT is thus an expansion in powers of p^2/v^2, with the normalization of the first term fixed since it is the GB kinetic term.

Now, what happens if the original theory becomes strongly coupled ($\lambda \geq 4\pi$)? In this case, we can no longer calculate the a_i's. However, as long as the $U(1)$ is still spontaneously broken, the EFT still has the same form, a derivative expansion, with the coefficient of the kinetic term fixed. Even though the theory is now strongly interacting, the Goldstone bosons are still weakly coupled at low energies, with their interactions suppressed by p^4/v^4, just by dimensional analysis. Thus, the most general EFT for Ξ is obtained by writing down the most general interactions invariant under the *unbroken* symmetry, with the size of the coefficients set by dimensional analysis in $1/v$.

Even in a simple spontaneously broken $U(1)$ theory, surprising powerful results may be obtained with little or no work. For a nice example of this this, the reader is referred to Weinberg's book[18], in which the salient features of superconductors are obtained in the EFT language, solely from power counting and the properties of the Goldstone bosons of a spontaneously broken $U(1)$.

Now let us extend this analysis to a more complicated broken symmetry group. In any low energy EFT of Goldstone bosons, the fields are constrained to lie on the vacuum manifold, since all other excitations have been integrated out of the theory. For example, extending this example to a theory of N fields, $\vec{\varphi} = (\varphi_1, \ldots, \varphi_N)$, the Lagrangian

$$\mathcal{L} = \sum_{i=1}^{N} \frac{1}{2}\partial_\mu \varphi_i \partial^\mu \varphi_i - \frac{\lambda}{4}\left(\sum_{i=1}^{N}\varphi_i^2 - v^2\right)^2 \tag{102}$$

has an $O(N)$ symmetry. If we choose the VEV of $\vec{\varphi}$ to lie in the φ_N direction, this breaks the $O(N)$ symmetry down to $O(N-1)$, the rotations of the $N-1$ other fields. There are $N-1$ broken generators, and so $N-1$ Goldstone bosons. Any point in the vacuum manifold may be obtained by acting on the VEV with a broken generator (moving along the bottom of the $N-1$ dimensional trough),

$$\vec{\varphi}\text{vacuum} = e^{i\sum_s t_s \lambda_s} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ v \end{pmatrix} \tag{103}$$

where the sum runs only over the *broken* generators t_s. Thus, we can describe the Goldstone modes by the field

$$\Xi = e^{i\sum_s t_s \pi_s(x)} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ v \end{pmatrix} \tag{104}$$

where the π_s's are Goldstone boson fields. Once again, the EFT may be expanded in powers of momentum,

$$\mathcal{L} = \frac{1}{2} \left[\partial_\mu \left(e^{-i\sum_s t_s \pi_s} \right) \partial^\mu \left(e^{i\sum_s t_s \pi_s} \right) \right]_{NN} + O(p^4) \tag{105}$$

(where the subscript NN refers to the element of the $N \times N$ matrix). Now, however, the nonabelian nature of the group gives us more information. Expanding the kinetic terms in terms of the GB fields π_s, the two-derivative term contains an infinite number of Goldstone boson self interactions, the coefficients of which are fixed by the spontaneously broken symmetry. Thus, spontaneously broken nonabelian global symmetries relate processes with different numbers of Goldstone bosons. Corrections to these relations arise from higher dimension operators in the Lagrangian, so are suppressed by power of p^2 and are not important at sufficiently low energies. These various relations go under the name of "current algebra", and can be obtained in other ways, but in the EFT language they simply fall out of the formalism. Furthermore, the EFT gives us a systematic approach to parameterizing the corrections to these relations.

3.2 Spontaneous Symmetry Breaking in QCD

Let us now apply these concepts to QCD. QCD with massless u, d, s quarks is invariant under separate flavour SU(3) rotations for left and right-handed quark fields:

$$\mathcal{L} = \bar{\psi}_L i \not{D} \psi_L + \bar{\psi}_R i \not{D} \psi_R \tag{106}$$

where

$$\psi = \begin{pmatrix} u \\ d \\ s \end{pmatrix} \tag{107}$$

and the symmetries are

$$\begin{aligned} \psi_L &\to L\psi_L, \quad L = e^{il_a T_a} \\ \psi_R &\to R\psi_R, \quad R = e^{ir_a T_a}. \end{aligned} \tag{108}$$

If this symmetry were not spontaneously broken, one could perform an axial transformation and change a scalar meson to a pseudoscalar, or a vector to a pseudovector. Thus, mesons would come in degenerate parity doublets (at least approximately, since the symmetry is weakly broken by the light quark masses and charges). The pion and the $a_0(980)$ would be approximately degenerate, as would the $\rho(770)$ and $a_1(1260)$. Since this is clearly not the case, chiral symmetry must be spontaneously broken. There is no scalar field around to pick up a VEV; instead, the fermion bilinear $\bar{\psi}\psi$ obtains a VEV due to nonperturbative effects

$$\langle\bar{\psi}\psi\rangle = \langle\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L\rangle \sim \text{ 1 GeV}^3. \tag{109}$$

This breaks the chiral symmetry down to the diagonal subgroup $L = R$ (the usual SU(3) of flavour, denoted SU(3)$_V$)

$$\psi \to e^{i\lambda_a T_a}\psi. \tag{110}$$

The axial symmetry, $L = R^\dagger$, or

$$\psi \to e^{i\gamma_5\lambda_a T_a}\psi \tag{111}$$

is spontaneously broken. There are eight broken generators, and thus there will be eight Goldstone bosons. These have the correct quantum numbers to be the three pions, four kaons and the eta.

The Chiral Lagrangian

In analogy with the $U(1)$ case, we can write down an EFT of only the Goldstone bosons. The other excitations like the ρ's and ω's have been integrated out of the theory, just like $\tilde{\varphi}_1$, so the excitations are constrained to lie along the vacuum manifold. The vacuum now has eight flat directions, corresponding to the eight broken generators. As in the previous examples, we work in "angular" variables. Fluctuations along these flat directions can be described by the field

$$\Sigma = e^{2i\tilde{\pi}/f} \tag{112}$$

where f is some dimensionful parameter required to get the units right, and

$$\tilde{\pi} = \pi^a T^a = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta \end{pmatrix} \tag{113}$$

is the Goldstone boson matrix. Under an $SU(3)_L \times SU(3)_R$ transformation, Σ transforms in the same way as the bilinear $\bar{\psi}_R \psi_L$,

$$\Sigma \to L \Sigma R^\dagger \tag{114}$$

This seemingly arbitrary choice is only one of an infinite number of ways to represent the PGB fields. The others are all related via field redefinitions, and so give the same S-matrix elements[19]. The essential feature of this representation is that under the unbroken $\mathrm{SU}(3)_V$, the pions transform *linearly* in the adjoint representation,

$$L = R = e^{i\lambda_a T_a} \Rightarrow \tilde{\pi} \to e^{i\lambda_a T_a} \tilde{\pi} e^{-i\lambda_a T_a} \tag{115}$$

while they transform nonlinearly under the broken symmetry,

$$L = R^\dagger = e^{i\lambda_a T_a} \Rightarrow \pi_a \to \pi_a + f\lambda_a + O(\lambda^2). \tag{116}$$

This is analogous to the transformation (99) in the $U(1)$ theory.

Written in terms of Σ, the EFT must be invariant under the full symmetry group. However, since the vacuum of the theory is at $\pi_a = 0$, or

$$\Sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{117}$$

the symmetry is indeed spontaneously broken to $\mathrm{SU}(3)_V$. Since the theory is strongly interacting at the breaking scale, we can't match onto the low-energy theory in perturbation theory; however, as in the $O(N)$ case, we can still expand the Lagrangian in powers of derivatives. Since we are interested in the low-energy dynamics of the Goldstone bosons, we should be able to truncate it after a few terms.

In accordance with the general requirement for Goldstone bosons, there are no terms in the EFT with 0 derivatives, since $\Sigma^\dagger \Sigma = 1$; the Goldstone bosons must therefore be derivatively coupled, as required on general grounds. Lorentz invariance forbids any term with a single derivative since there is no other vector in the theory. At two derivatives, there is only one possible term,

$$\mathrm{Tr}\left[\partial_\mu \Sigma^\dagger \partial^\mu \Sigma\right] = \frac{4}{f^2} \mathrm{Tr}\left[\partial_\mu \tilde{\pi} \partial^\mu \tilde{\pi}\right] + \ldots \tag{118}$$

(any other two-derivative term may be put in this form after an integration by parts). Immediately we recognize this term as the kinetic term for the massless GB's. This term must be correctly normalized, so we have our effective Lagrangian to $O(p^2)$:

$$\mathcal{L}_{\mathrm{EFT}} = \frac{f^2}{4} \mathrm{Tr} \left[\partial_\mu \Sigma^\dagger \partial^\mu \Sigma \right] + O(p^4). \tag{119}$$

As in the $O(N)$ case discussed previously, the kinetic term contains an infinite number of multi-particle interactions, whose coefficients are all related by chiral symmetry. Expanding $\mathcal{L}$ in terms of the fields, we find

$$\mathcal{L}_{\mathrm{EFT}} = \mathrm{Tr} \partial_\mu \tilde{\pi} \partial^\mu \tilde{\pi} + \frac{1}{3f^2} \mathrm{Tr} \left[\tilde{\pi}, \partial_\mu \tilde{\pi} \right]^2 + \ldots \tag{120}$$

where the ellipses denote additional multi-pion interactions. Thus, to $O(p^2)$ all the PGB self interactions are determined by a *single* constant f.

Symmetry Currents

The only constant we have to determine in $\mathcal{L}$ to this order is f. This can be related to an observable in a very elegant way. Recall that a symmetry of a theory corresponds to a conserved current. In QCD, the currents associated with SU(3) $\times$ SU(3) are easy to calculate, via the standard Noether procedure. They are

$$j^\mu_{Aa} = \bar{\psi} \gamma^\mu \gamma_5 T_a \psi, \quad j^\mu_{Va} = \bar{\psi} \gamma^\mu T_a \psi \tag{121}$$

where A and V refer to axial and vector transformations, respectively. The interesting observation is that these currents *also* appear in the weak Hamiltonian describing, for example, pion decay,

$$\mathcal{H}_W = \frac{c}{m_W^2} \bar{u} \gamma^\mu (1 - \gamma_5) d \bar{\nu}_e \gamma_\mu (1 - \gamma_5) e + \ldots \tag{122}$$

even though in this context they have nothing to do with the global SU(3) $\times$ SU(3) symmetry (instead, they are weak gauge symmetry currents). But a current is a current, and since the EFT has the same symmetries as full QCD, the conserved currents in the two theories must be the same. (This is the same situation that arose in HQET, where we exploited the fact that weak currents were global symmetry currents in the EFT to determine their matrix elements.) The matrix element of the weak current between a pion and the vacuum is measured in pion decay,

$$\langle 0 | j^{\mu a}_A | \pi^b \rangle \equiv i f_\pi p^\mu \delta^{ab} \tag{123}$$

where $f_\pi \simeq 93$ MeV is the pion decay constant. Thus, if we construct the symmetry current corresponding to an axial flavour rotation in the effective theory, we can calculate its matrix element between a pion and the vacuum; this will allow us to determine f in terms of f_π.

Recall the Noether procedure: if $\mathcal{L}$ is invariant under an infinitesimal global symmetry transformation with parameter ϵ, the corresponding conserved current is given by the change in $\mathcal{L}$ when ϵ is taken as a function of x:

$$\delta\mathcal{L} = \partial^\mu \epsilon(x) j^\mu(x). \tag{124}$$

Let us now construct the symmetry currents in the effective theory. Under an infinitesimal LH transformation, we have

$$\Sigma \to \Sigma + i\epsilon_L^a T^a \Sigma. \tag{125}$$

Substituting this into Eq. (119), gives

$$\delta\mathcal{L} = \partial_\mu \epsilon_L^a \mathrm{Tr}\left[T^a \Sigma \partial^\mu \Sigma^\dagger\right] \tag{126}$$

and so the conserved current corresponding to an $\mathrm{SU}(3)_L$ transformation is

$$j_L^{\mu a} = \frac{i}{2} f^2 \mathrm{Tr}\left[T^a \Sigma \partial^\mu \Sigma^\dagger\right]. \tag{127}$$

Similarly,

$$j_R^{\mu a} = \frac{i}{2} f^2 \mathrm{Tr}\left[T^a \Sigma^\dagger \partial^\mu \Sigma\right] \tag{128}$$

and so the axial current is

$$j_A^{\mu a} = j_R^{\mu a} - j_L^{\mu a} = -f\partial^\mu \tilde{\pi} + \ldots \tag{129}$$

where we have expanded the Σ field in terms of GB fields, and the dots denote terms with additional fields.

This is worth contemplating for a moment - using the symmetries of the theory, we have expressed a *quark* current in terms of *pion* fields! This is real nonperturbative information, and we got it without solving anything nonperturbative. Such is the power of symmetry. Now, matrix elements of $\tilde{\pi}$ fields are trivial to calculate,

$$\langle 0|j^{\mu a}|\pi^b\rangle = \langle 0| - f\partial^\mu \tilde{\pi}^a + \ldots |\pi^b\rangle = ifp^\mu \delta^{ab}. \tag{130}$$

Comparing this to Eq. (123) we immediately conclude

$$f = f_\pi \simeq 93 \text{ MeV}. \tag{131}$$

Thus, we have completely determined $\mathcal{L}$ to $O(p^2)$.

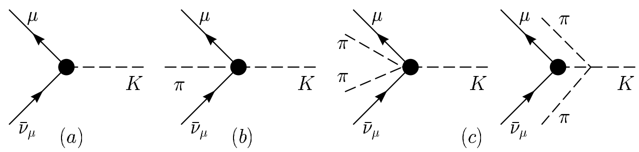

Figure 14: Diagrams contributing to semileptonic K decay to leading order: (a) $K \to \mu\bar{\nu}_\mu$, (b) $K \to \pi\mu\bar{\nu}_\mu$, (c) $K \to \pi\pi\mu\bar{\nu}_\mu$.

Application: Semileptonic K Decay

Already we have a lot of predictive power, particularly for semileptonic K decays. This is because the decays

$$K^- \to \mu\nu, \quad K^- \to \mu\nu\pi^0, \quad K^- \to \mu\nu\pi^+\pi^-, \ldots \tag{132}$$

are all determined by matrix elements of the weak current $\bar{u}\gamma^\mu(1-\gamma_5)s$, which is also a flavour symmetry current of the chiral Lagrangian.

By arguments identical to those in the last section, this current may be written in terms of the Goldstone boson fields. Carrying out the expansion to a few more orders in the fields gives

$$\begin{aligned} \bar{u}\gamma^\mu(1-\gamma_5)s \to & -\sqrt{2}f\partial^\mu K^- - \frac{i}{\sqrt{2}}\left[K^-\partial_\mu\pi^0 - \pi^0\partial_\mu K^-\right] \\ & +\frac{\sqrt{2}}{3f}\left[K^-\pi^+\partial_\mu\pi^- - 2K^-\pi^-\partial_\mu\pi^+ - \frac{1}{2}K^-\pi^0\partial_\mu\pi^0 \right. \\ & \left. +\frac{1}{2}\pi^0\pi^0\partial_\mu K^- + \pi^+\pi^-\partial_\mu K^-\right] + \ldots . \end{aligned} \tag{133}$$

Thus, we can compute all these decay rates in chiral perturbation theory, solely in terms of f. The relevant diagrams are shown in Fig. 14.

3.3 The Chiral Symmetry Breaking Scale

So far, our results have been determined only by symmetries of the theory, and have been independent of the details of QCD (and thus, the success of these predictions tells us no more about QCD than its global symmetry structure). The deviations from these predictions arise at higher orders in p^2. At $O(p^4)$, for example, there are four derivative terms like terms like

$$\mathcal{L}_4 = a_1 \mathrm{Tr}\partial_\mu\Sigma\partial^\mu\Sigma^\dagger\partial_\nu\Sigma\partial^\nu\Sigma^\dagger + \ldots . \tag{134}$$

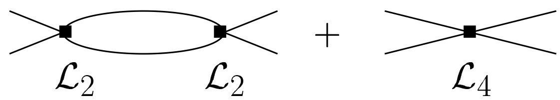

Figure 15: $\pi - \pi$ scattering at $O(p^4)$.

By dimensional analysis, this term will give a contribution of order p^2/Λ_χ^2 relative to the leading term, where Λ_χ is the scale of new physics at which the EFT breaks down, set by the scale of the VEV of $\bar{\psi}\psi$. If QCD were weakly interacting at the matching scale we could calculate a_1, but since it is not, all we have is dimensional analysis. Physically, we expect $\Lambda_\chi \sim m_\rho$, since this is the lightest resonance we have integrated out of the theory. However, we can say something about a_1 without knowing anything about the higher resonances.

The only dimensionful parameter in the chiral Lagrangian at leading order is $f_\pi \sim 93$ MeV, and the nonrenormalizable terms in Eq. (120) are suppressed by powers of $1/f_\pi$, suggesting that $\Lambda_\chi \sim f_\pi$. If this were indeed the case, chiral symmetry would be completely irrelevant in the real world, since the EFT would break down at a scale smaller than the pion mass. However, as we will see, this is not necessarily the case - there are important factors of 4π in any estimate of the scale where the theory breaks down which save us. The art of estimating the relevant factors of 4π goes under the name of "naïve dimensional analysis" (NDA)[20].

Consider $\pi - \pi$ scattering at $O(p^4)$. At this order, there are contributions from both $\mathcal{L}_4$, and from a loop containing two operators in $\mathcal{L}_2$, as shown in Fig. 15. Indeed, the operators in $\mathcal{L}_4$ are required as counterterms for the divergent loop diagram in Fig. 15. Thus, the separation of the amplitude into the two separate graphs is renormalization scheme-dependent, and therefore arbitrary. In fact, it requires a fine-tuning to make the contribution of $\mathcal{L}_4$ much smaller than that of the first diagram.

More precisely, the first graph in Fig. 15 is proportional to

$$\int \frac{d^4p}{(2\pi)^4}\frac{p^2}{f^2}\frac{p^2}{f^2}\frac{1}{p^4} \sim \frac{p^4}{16\pi^2}\frac{1}{f^4}\log\mu \tag{135}$$

where we have regulated the divergence in $\overline{\text{MS}}$. Since the sum of the two graphs is μ independent (the coefficients in $\mathcal{L}_4$ depend on μ), changing μ by

$O(1)$ changes the coefficients of operators in $\mathcal{L}_4$ by order

$$\frac{p^4}{16\pi^2}\frac{1}{f^4}. \tag{136}$$

Thus, even if the contribution of $\mathcal{L}_4$ is much smaller than this value at one value of μ, at an equally good value of μ it is of the same order. Thus, in Eq. (134), naturalness implies than

$$a_1 \geq \frac{1}{16\pi^2} \tag{137}$$

and so we have

$$\Lambda_\chi \leq 4\pi f_\pi \sim 1 \text{ GeV}. \tag{138}$$

The theory would have to be fine-tuned if in the $\overline{\text{MS}}$ scheme, the coefficients in $\mathcal{L}_4$ were significantly smaller than this estimate.

Thus, the estimate $\Lambda_\chi \sim f_\pi$ is unduly pessimistic. Although the scale at which the theory breaks down is indeed set by f_π, it may be significantly higher (although it cannot naturally be arbitrarily high). This idea of NDA is that this inequality is actually a rough equality - that is, hope that we are lucky and that Λ_χ is no smaller than its natural size. Thus, in general we expect the coefficients of n-derivative terms in $\mathcal{L}$ to be

$$f_\pi^2 \times \frac{O(1)}{\Lambda_\chi^{n-2}}. \tag{139}$$

Experimentally, this works pretty well. Many of the four-derivative terms in the chiral Lagrangian have been fit to experiment by Gasser and Leutwyler[17]. NDA tells us the natural size of these coefficients is

$$\frac{f_\pi^2}{\Lambda_\chi^2} = \frac{1}{16\pi^2} \sim 7 \times 10^{-3}. \tag{140}$$

Those that are measured vary between 0.4×10^{-3} and 7.4×10^{-3}, so NDA certainly seems to be in the right ballpark (which is all we can require of it).

3.4 Explicit Symmetry Breaking

Of course, chiral SU(3)×SU(3) is not an exact symmetry of QCD. It is explicitly broken by both the quark masses,

$$\mathcal{L}_m = \bar{\psi}_L M \psi_R + \text{h.c.}, \tag{141}$$

where M is the quark mass matrix,

$$\begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}, \tag{142}$$

and by the quark electromagnetic couplings. However, this breaking is small (in a sense which we shall define shortly), and these corrections may be taken into account perturbatively. Since the symmetry is not exact, the Goldstone bosons will not be exactly massless. Such fields are known as pseudo-Goldstone bosons (PGB's). Since the breaking due to the s quark mass is by far the largest of the symmetry breaking effects, let us concentrate on the quark masses. To consistently take these effects into account, we make use of a very nice trick. Suppose for a moment that the quark mass matrix M were actually a classical external field (sometimes called a "spurion" field) transforming under chiral rotations as

$$M \to LMR^\dagger. \tag{143}$$

In this case the mass term (141) would not break chiral symmetry. However, the chiral Lagrangian would have to include all chirally invariant terms coupling M and Σ, with unknown coefficients; for example

$$\mathrm{Tr}\Sigma^\dagger M + h.c.\ . \tag{144}$$

We can recover QCD by taking M to be a constant,

$$\begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}. \tag{145}$$

The interactions between M and Σ then determine the effects of the quark masses in the low-energy theory. Furthermore, since M is small ($m_u,\ m_d, m_s \ll \Lambda_\chi$), the effective theory may be expanded in powers of M, and truncated after the first term or so. At leading order in M/Λ_χ there is only one term

$$\mathcal{L}_M = \frac{1}{2}\mu f_\pi^2 \mathrm{Tr}\Sigma^\dagger M + \mathrm{h.c.} \tag{146}$$

(note that this is not a derivative interaction, since it explicitly breaks chiral symmetry) where μ is an unknown parameter with dimension of mass. Expanding Eq. (146) in terms of the PGB's gives

$$\mathcal{L}_M = -2\mu \mathrm{Tr} M \tilde{\pi}^2 + \dots \tag{147}$$

which we immediately recognize as a mass term for the PGB's, and from which we can read off the masses:

$$\begin{aligned} m^2_{\pi^\pm} &= \mu(m_u + m_d) \\ m^2_{K^\pm} &= \mu(m_u + m_s) \\ m^2_{K^0,\ \bar{K^0}} &= \mu(m_u + m_d) \end{aligned} \tag{148}$$

and the $\pi^0,\ \eta$ mass-squared matrix is

$$\mu \begin{pmatrix} m_u + m_d & \frac{m_u - m_d}{\sqrt{3}} \\ \frac{m_u - m_d}{\sqrt{3}} & \frac{1}{3}(m_u + m_d + 4m_s) \end{pmatrix}. \tag{149}$$

Ignoring the off-diagonal term, whose effects are second order in the isospin-violating mass different $m_u - m_d$, this gives

$$\begin{aligned} m^2_{\pi^0} &= \mu(m_u + m_d) \\ m^2_\eta &= \frac{\mu}{3}(4m_s + m_u + m_d). \end{aligned} \tag{150}$$

Thus, we can relate the meson masses to the underlying quark masses. The meson masses satisfy the Gell-Mann Okubo relation

$$3m^2_\eta + m^2_\pi = 4m^2_K. \tag{151}$$

Note, however, that since masses occur in the combination μm, we cannot use this approach to determine absolute quark masses, only ratios. The electromagnetic contribution to the masses of the charged kaons is the same as that for the pions, so the electromagnetic effects drop out of the following ratios

$$\begin{aligned} \frac{m_u}{m_d} &\simeq \frac{m^2_{K^+} - m^2_{K^0} + 2m^2_{\pi^0} - m^2_{\pi^+}}{m^2_{K^0} - m^2_{K^+} + m^2_{\pi^+}} \simeq 0.55 \\ \frac{m_s}{m_d} &\simeq \frac{m^2_{K^0} + m^2_{K^+} - m^2_{\pi^+}}{m^2_{K^0} - m^2_{K^+} + m^2_{\pi^+}} \simeq 20.1. \end{aligned} \tag{152}$$

(Note, however, that the first ratio is very sensitive to higher order corrections, and so cannot be taken as an unambiguous sign that $m_u \neq 0$.)

This type of analysis can be extended to other symmetry breaking terms in $\mathcal{L}$. For example, the $\Delta S = 1$ nonleptonic operator

$$\bar{d}\gamma^\mu(1 - \gamma_5)s\bar{u}\gamma_\mu(1 - \gamma_5)u \tag{153}$$

(written in this form after a Fierz transformation) may be decomposed into terms which transform (in the same sense as the mass term is said to transform) as 8 and 27 dimensional representations of $SU(3)_L$. Decomposing these terms further in terms of isospin, the first is pure $\Delta I = 1/2$, while the second contains both $\Delta I = 1/2$ and $\Delta I = 3/2$ terms. Experimentally, $\Delta I = 1/2$ transitions in nonleptonic K decay are much larger than $\Delta I = 3/2$, so to a good approximation the decay is pure octet, and for simplicity we will ignore the **27**.

The pure octet piece of the $\Delta S = 1$ operator is

$$\bar{\psi} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \gamma^\mu (1-\gamma_5)\psi\bar{\psi}\gamma_\mu(1-\gamma_5)\psi \tag{154}$$

and so we can introduce the spurion field

$$h = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{155}$$

which "transforms" under the chiral symmetry as $h \to LhL^\dagger$. At leading order in h, there is only one term which couples h to Σ,

$$\mathcal{L}_{\Delta S=1} = \frac{1}{4} f_\pi^2 \lambda \mathrm{Tr}\left[(h+h^\dagger)\partial_\mu \Sigma \partial^\mu \Sigma^\dagger\right] \tag{156}$$

where λ is an unknown constant. However, since this is the operator responsible for $K_S \to \pi\pi$ decays, we can measure it:

$$\lambda = 3.2 \times 10^{-7}. \tag{157}$$

The EFT may now be used to calculate other weak decays mediated by the same operator; we will see an example in the next section.

3.5 Applications

In this section I will give a couple of examples of non-trivial calculations in chiral perturbation theory. I won't give a completely detailed description of all the ingredients, but I hope that they will give you a sense of the structure of such calculations. For more details, I refer you to the original papers, cited in the references.

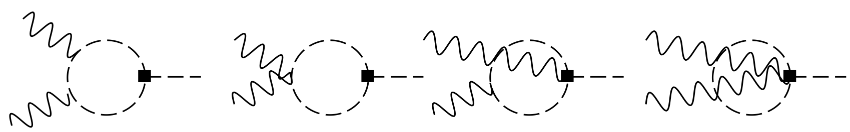

Figure 16: Diagrams contributing to $K_S \to \gamma\gamma$. The box denotes the octet $\Delta S = 1$ operator.

$K_s \to \gamma\gamma$

This process was calculated in [21]. The new ingredient in this calculation is the addition of electromagnetism, and the decay proceeds through the graphs shown in Fig. 16. The PGB-photon vertices just arise from making the derivatives in Eq. (119) covariant,

$$D_\mu \Sigma = \partial_\mu \Sigma + ieA_\mu \left[Q, \Sigma\right] \tag{158}$$

where Q is the quark charge matrix (again, a spurion field)

$$Q = \frac{1}{3}\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{159}$$

Similarly, the extra photons coming off the $\Delta I = 1/2$ operator in Eq. (156) arise from the covariant derivatives in

$$\mathcal{L}_{\Delta I=1/2} = \frac{1}{4} f_\pi^2 \lambda \mathrm{Tr}(h + h^\dagger) D_\mu \Sigma D^\mu \Sigma^\dagger. \tag{160}$$

The loop integrals are convergent, and there is no local operator at this order which contributes to the decay. The result is

$$\mathrm{Br}(K_s \to \gamma\gamma) = 2.4 \times 10^{-6} \tag{161}$$

which compares nicely with the experimental measurement of $(2.4 \pm 0.9) \times 10^{-6}$.

$\eta(\pi^0) \to \ell^+\ell^-$

This process has been considered in a number of papers, most recently and completely in [22]. The extra ingredient required for this calculation is the axial anomaly. In QCD, the axial current

$$j^{\mu 3} \equiv \bar{\psi}\gamma^\mu \gamma_5 T^3 \psi \tag{162}$$

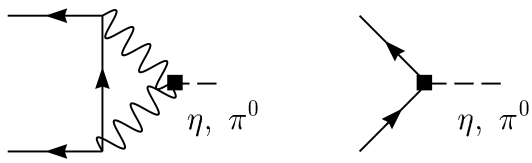

Figure 17: Diagrams contributing to $\eta, \pi^0 \to \ell^+\ell^-$.

associated with the π^0 and η is not only spontaneously broken, but also explicitly broken by electromagnetic effects, through the famous triangle anomaly. By calculating the triangle graph, one find that the current is not conserved, but rather satisfies

$$\partial_\mu j_A^{\mu 3} = \frac{e^2}{48\pi^2}\epsilon_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta}. \tag{163}$$

To reproduce this in the chiral Lagrangian, an explicit symmetry breaking term must be added to the theory with the property that under a T_3 chiral transformation this term is not invariant, but rather changes by the appropriate factor. This is known as the Wess-Zumino term, and at leading order in the fields, it is

$$\mathcal{L}_{WZ} = \frac{\alpha}{4\pi f}\epsilon_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta}\left(\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}}\right) + \dots \tag{164}$$

and is the operator responsible for $\pi^0 \to \gamma\gamma$ and $\eta \to \gamma\gamma$ decays. Note that there are no unknown parameters in this term; its coefficient is fixed by requiring that it reproduce the anomaly in the full theory.

The Wess-Zumino term contributes to the decay $\eta \to \mu^+\mu^-$ via the first graph in Fig. 17. The loop graph is UV divergent, so it clearly needs a counterterm, which gives the second diagram. Alternatively, one can just notice that one has to include any local operator with the correct quantum numbers at the same order in the chiral expansion. It turns out that there are two such operators,

$$\begin{aligned}\mathcal{L}_{\text{c.t.}} = \frac{3i\alpha^2}{32\pi^2}\bar{\ell}\gamma^\mu\gamma_5\ell\ \big[&\chi_1 Tr(Q^2\Sigma^\dagger\partial_\mu\Sigma - Q^2\partial_\mu\Sigma^\dagger\Sigma) \\ &+\chi_2 Tr(Q\Sigma^\dagger Q\partial_\mu\Sigma - Q\partial_\mu\Sigma^\dagger Q\Sigma)\big]\end{aligned} \tag{165}$$

where χ_1 and χ_2 are unknown coefficients. Note that since the decay occurs via two photons, the counterterm has two powers of Q. Using the measured branching ratio

$$\text{Br}(\eta \to \mu^+\mu^-) = 5.8 \pm 0.8 \times 10^{-6} \tag{166}$$

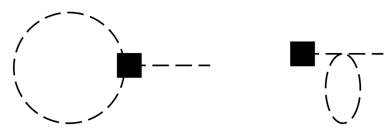

Figure 18: Leading SU(3) breaking corrections to f_K/f_π. The box indicates an insertion of the weak current; the leptons are omitted.

allows us to fit the sum of these unknown coefficients,

$$-40 < \chi_1(\mu = 1 \text{ GeV}) + \chi_2(\mu = 1 \text{ GeV}) < -13. \tag{167}$$

(Note that this is consistent with the NDA expectation $3\chi_1/32 \sim 1$.) This now gives us predictions for $\eta \to e^+e^-$ and $\pi^0 \to e^+e^-$, which depend on the same linear combination of counterterms. Note that this is not a simple SU(3)-like prediction; the loop integrals are weighted in different parts of momentum space. The predictions are

$$\begin{aligned} \text{Br}(\pi^0 \to e^+e^-) &= 7 \pm 1 \times 10^{-8} \quad (\text{exp't:} \;\; 7.5 \pm .2 \times 10^{-8}) \\ \text{Br}(\eta \to e^+e^-) &= 5 \pm 1 \times 10^{-9} \quad (\text{exp't:} \;\; < 2 \times 10^{-4}). \end{aligned} \tag{168}$$

3.6 Other Applications

I hope I have given you a flavour of the applications of chiral perturbation theory here. Let me just mention a few more of the applications.

Chiral Logarithms: SU(3) breaking effects from local operators give terms proportional to m_q, or $m^2_{\pi,K,\eta}$. Since they are local operators, these must be analytic in m_q. Contributions which are nonanalytic in the mesons masses, for example $\sim m_\pi^2 \log m_\pi^2$, can only arise from loop integrals, which are sensitive to infrared effects. Thus, in the chiral limit $m_q \to 0$, these terms formally dominate over the nonanalytic terms. Furthermore, they are calculable. Of course, since $\log m_\pi^2/\Lambda_\chi^2 \sim -3.9$ and $\log m_K^2/\Lambda_\chi^2 \sim -1.4$, these terms are not greatly enhanced. Nonetheless, one often calculates these terms to at least set a lower limit on the size of SU(3) breaking effects. One oft-cited example is the ratio of decay constants f_K/f_π. At one loop, this is determined by the graphs in Fig. 18, which yield (ignoring the pion mass)

$$\begin{aligned} \frac{f_K}{f_\pi} &= 1 - \frac{3m_K^2}{64\pi^2 f_\pi^2} \log m_K^2/\Lambda_\chi^2 + O(m_K^2) \\ &= 1.19. \end{aligned} \tag{169}$$

This is rather close (probably too close to be other than luck) to the experimental value of 1.2.

Matter Fields: It is problematic to introduce heavy particles (vector mesons, nucleons, heavy hadrons) into chiral perturbation theory in the standard way, since their masses of order, or greater than, Λ_χ. However, for massive stable particles, this energy cannot be released, and using techniques similar to HQET they can be introduced as static chirally coupled fields in an SU(3) invariant manner. This gives an effective field theory describing low energy πp, πD, etc., interactions. There are a number of uses of such a theory, such as nucleon-pion scattering near threshold, $D^* \to D\pi$ and $D^* \to D\gamma$ decays, nonanalytic SU(3) breaking in heavy particle masses and couplings, and decays of excited heavy states such as $D_2^* \to D\pi$. For some examples, see [23].

Nuclear Forces: There has recently been a great deal of excitement about the possibility of studying nuclear forces (that is, nucleon-nucleon scattering and beyond) using a chiral Lagrangian. This would provide a model-independent approach to many classic nuclear physics problems, such as the internucleon potential, properties of the deuteron, and properties of bulk nuclear matter. Until quite recently the major stumbling block was coming up with a consistent power counting scheme, which is significantly more difficult than in the case of the standard chiral Lagrangian, due to the apparent fine-tuning in nature of the NN phase shift and deuteron binding energies. Recently, this problem seems to have been dealt with consistently, and there has been much activity in this field [24].

Electroweak Symmetry Breaking: Finally, we recall that the chiral Lagrangian approach was based solely on the symmetries, and so the same approach is appropriate to study any theory exhibiting chiral symmetry breaking. In particular, the Standard Model has the symmetry-breaking pattern

$$\mathrm{SU}(2)_{EW} \times \mathrm{SU}(2)_R \to \mathrm{SU}(2)_{\mathrm{custodial}} \tag{170}$$

where the $\mathrm{SU}(2)_R$ is a global symmetry with a gauged $U(1)$, and the remaining "custodial" SU(2) guarantees that the ρ parameter is near one. Thus, one can use chiral perturbation theory techniques to describe the low-energy effective theory of the symmetry-breaking sector. This is particularly useful in strongly-coupled theories with dynamical symmetry breaking, such as technicolour and its generalizations. The leading terms in the chiral Lagrangian reproduce the tree-level results of the Standard Model, while deviations from the SM arise from higher dimension operators. Naïve dimensional analysis also provides a useful means of estimating the effects of a strongly interacting symmetry breaking sector [25].

Acknowledgments

I would like to thank the organizers for inviting me to lecture at the Institute, and for their hospitality during the workshop. This work was supported in part by a grant from the Natural Sciences and Engineering Research Council of Canada, and by the Sloan Foundation.

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This list of references is by no means exhaustive; I have only referenced some useful reviews on the subject, as well as the original literature for specific points made in the text.

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